Numerical modeling of rock deformation:

02 Dimensional analysis

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AS 2009, Thursday 10-12, NO D 11
Introduction

Why doing dimensional analysis?

• Converting units
• Checking equations
• First order estimate of results
• Reducing number of parameters
• Determine controlling parameter
• Determine relative importance of parameters
Units

- We can use four fundamental units:
  * **Length** (*L*),
  * **Time** (*T*),
  * **Mass** (*M*) and
  * **Temperature** (*θ*)
- Always know the units of the parameters you use!

<table>
<thead>
<tr>
<th>Name</th>
<th>SI units</th>
<th>Basic units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time</td>
<td>[s]</td>
<td>T</td>
</tr>
<tr>
<td>Length</td>
<td>[m]</td>
<td>L</td>
</tr>
<tr>
<td>Mass</td>
<td>[kg]</td>
<td>M</td>
</tr>
<tr>
<td>Velocity</td>
<td>[m s⁻¹]</td>
<td>LT⁻¹</td>
</tr>
<tr>
<td>Acceleration</td>
<td>[m s⁻²]</td>
<td>LT⁻²</td>
</tr>
<tr>
<td>Density</td>
<td>[kg m⁻³]</td>
<td>ML⁻³</td>
</tr>
<tr>
<td>Force</td>
<td>[N], [kg m s⁻²]</td>
<td>MLT⁻²</td>
</tr>
<tr>
<td>Stress</td>
<td>[Pa], [N m⁻²]</td>
<td>ML⁻¹T⁻²</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>[Pa]</td>
<td>ML⁻¹T⁻²</td>
</tr>
<tr>
<td>Viscosity</td>
<td>[Pa s]</td>
<td>ML⁻¹T⁻¹</td>
</tr>
<tr>
<td>Energy</td>
<td>[N m], [J]</td>
<td>ML²T⁻²</td>
</tr>
<tr>
<td>Power</td>
<td>[J s⁻¹], [W]</td>
<td>ML²T⁻³</td>
</tr>
<tr>
<td>Strain rate</td>
<td>[s⁻¹]</td>
<td>T⁻¹</td>
</tr>
<tr>
<td>Temperature</td>
<td>[K]</td>
<td>0</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>[W m⁻¹ K⁻¹]</td>
<td>ML⁻³θ⁻¹</td>
</tr>
<tr>
<td>Specific heat</td>
<td>[J kg⁻¹ K⁻¹]</td>
<td>L²T⁻²θ⁻¹</td>
</tr>
</tbody>
</table>
Units

Party planning: You invite 30 friends and calculate 4 beers for each friend. How many sixpacks do you have to buy?

\[ 30 \text{[friend]} \cdot 4 \left[ \frac{\text{beers}}{\text{friend}} \right] \cdot 6 \left[ \frac{\text{beers}}{\text{sixpack}} \right] = 20 \left[ \text{sixpack} \right] \]

The viscosity, \([\text{Pa s}]\), of rocks is a function of several parameters. What are the units of \(R\), \(V\) and \(A\)?

\[ \mu = \varepsilon^n \frac{1}{A^n} \exp \left( \frac{E + V}{nRT} \right) \]

\[ [T] = K, [E] = \frac{J}{\text{mol}}, [\varepsilon] = \frac{1}{s} \]

\[ [\mu] = [\varepsilon]^{n-1} [A]^n \]

\[ Pa \ s = \left( \frac{1}{s} \right)^n \left( \frac{1}{s} \right)^{-1} \left( ? \right)^{\frac{1}{n}} = \left( \frac{1}{s} \right)^n (?)^\frac{1}{n} \Rightarrow (?)^\frac{1}{n} = \text{Pa}^n s \]

E and V must have the same units, because they are added. 
\(n\) must be dimensionless, because it is an exponent. 
The ratio \(E/(nRT)\) is dimensionless since the exponential function is dimensionless.
Dimensionless Functions

The following functions and quantities have no dimension:

• Sine, cosine, etc. Functions
• Exponential functions
• Logarithms
• Angles
Atomic explosion

The British physicist G.I. Taylor estimated the energy of the first atomic explosion in 1945 based on a series of pictures that were published in a popular magazine. At that time the energy of the atomic explosion was considered top secret and Taylor’s estimate caused “much embarrassment” in American government circles, because the series of pictures was not classified.

\[
[E] = ML^2T^{-2}, [t] = T, [\rho] = ML^{-3}
\]
\[
[R] = \left[ E \right]^{\frac{1}{5}} \left[ t \right]^{\frac{2}{5}} \left[ \rho \right]^{-\frac{1}{5}}
\]
\[
\Pi = \frac{R}{E^{\frac{1}{5}} t^{\frac{2}{5}} \rho^{\frac{1}{5}}} = \text{const}, \quad \text{assume } \text{const} \approx 1
\]
\[
E = \frac{R^5 \rho}{t^2} \approx \frac{80^{5.2}}{0.006^2} = 10^{14} \approx 25 \text{ kilo tons of TNT}
\]

Dimensional analysis is a powerful tool!
The pendulum 1

What is the period of the pendulum?

On what can the period of oscillation depend?

• Length of pendulum
• Mass of bob
• Gravitational acceleration

Length: L (e.g., meter)
Time: T (e.g., second)
Mass: M (e.g., kilogram)

The quantity Π is dimensionless and does not change when the fundamental units of measurements are changed.

\[ ml \frac{\partial^2 \alpha(t)}{\partial t^2} + mg \alpha(t) = 0 \]

Period of oscillation:

\[ \theta = 2\pi \sqrt{\frac{l}{g}} \]

\[ \Pi = \frac{\theta}{\sqrt{\frac{l}{g}}} = \text{const} \]

The quantity Π is dimensionless and does not change when the fundamental units of measurements are changed.
The pendulum 2

On what can the period of oscillation depend?

\[ \theta = f(l, m, g) \]

\[ [l] = L \]

\[ [m] = M \]

\[ [g] = \frac{L}{T^2} \]

Dimensionless form 3 independent units

\[ \frac{\theta}{[\theta]} = f\left(\frac{l}{[l]}, \frac{m}{[m]}, \frac{g}{[g]}\right) \]

\[ [\theta] = T = [l]^x [m]^y [g]^z = L^x M^y \left(\frac{L}{T^2}\right)^z \]

\[ x = \frac{1}{2}, y = 0, z = -\frac{1}{2} \]

\[ [\theta] = L^2 M^0 \left(\frac{L}{T^2}\right)^{-\frac{1}{2}} = [l]^{\frac{1}{2}} [g]^{\frac{1}{2}} = \sqrt{\frac{l}{g}} \]

\[ \frac{\theta}{\sqrt{\frac{l}{g}}} = f(1, 1, 1) = \text{const} \]

\[ \theta = \text{const} \sqrt{\frac{l}{g}} \]
What is the characteristic wavelength of viscous folds?

Analytical solution:

$$
\lambda = 2\pi 6^{\frac{1}{3}} H \left( \frac{\mu_1}{\mu_2} \right)^{\frac{1}{3}}
$$

$$
const = 2\pi 6^{\frac{1}{3}}
$$

$$
f = \left( \frac{1}{3} \right)
$$
Folding wavelength

$$\lambda = f(H, P, \mu_1, \mu_2)$$

**Note:**
The relation between the dominant wavelength and the viscosity ratio can be determined with only one set of experiments only varying the viscosity ratio. Dummies would perform four sets of experiments varying individually the parameters $H$, $P$, $\mu_1$ and $\mu_2$.

Analytical solution:

$$\lambda = 2\pi 6^{-\frac{1}{3}} H \left( \frac{\mu_1}{\mu_2} \right)^{\frac{1}{3}}$$

$$\text{const} = 2\pi 6^{-\frac{1}{3}}$$

$$f = \left( \frac{1}{3} \right)$$
Controlling parameters

\[
\frac{\mu H^3}{3} \left( \frac{\partial^5}{\partial x^4 \partial t} W(x, t) \right) + 4 \mu e H \left( \frac{\partial^2}{\partial x^2} W(x, t) \right) + 2 \eta \omega \left( \frac{\partial}{\partial t} W(x, t) \right) + \Delta \rho g W(x, t) = 0
\]

\[W(x, t) := A0 e^{(\omega x I + \alpha t)}\]

\[
\alpha := \frac{3 n (4 \mu e H \omega^2 - \Delta \rho g)}{\omega (\mu H^3 + 3 + 6 \eta n)}
\]

\[
\alpha_s := \frac{54 (2 k^2 - 1)}{k (9 k^3 + 4 S \sqrt{6})}
\]

\[\alpha_s = \frac{\alpha A_r}{n e} \quad S = \frac{9 \sqrt{6} \eta n}{\left(\frac{3}{2}\right)} \quad A_r = \frac{\Delta \rho g H}{2 \mu e} \quad k = \frac{\omega H}{\sqrt{A_r}}\]

7 parameters: n, μ, e, H, Δρ, g, η

1 parameter: S

Schmalholz et al., JGR, 2002
Literature

- Scaling, self-similarity and intermediate asymptotics. Barenblatt, G.I.
- Paper on course web page, (Schmalholz et al., JGR, 2002)
- PDF script on course web page