Finite amplitude folding: transition from exponential to layer length controlled growth

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Abstract

A new finite amplitude theory of folding has been developed by the combined application of analytical, asymptotic and numerical methods. The existing linear folding theory has been improved by considering nonlinear weakening of membrane stresses, which is caused by the stretching of the competent layer during folding. The resulting theory is simple and accurate for finite amplitude folding and is not restricted to infinitesimal amplitudes, as is the classical linear theory of folding. Two folding modes relevant to most natural settings were considered: (i) both membrane and fiber stresses are viscous during folding (the ‘viscous’ mode); (ii) membrane stresses are viscous whereas fiber stresses are elastic (the ‘viscoelastic’ mode). For these two modes, the new theory provided a nonlinear, ordinary differential equation for fold amplification during shortening and an estimate for crossover amplitude and strain where the linear theory breaks down. A new analytical relationship for amplitude versus strain was derived for strains much larger than the crossover strain. The new relationship agrees well with complete 2D numerical solutions for up to threefold shortening, whereas the exponential solution predicted by the linear theory is inaccurate by orders of magnitude for strains larger than the crossover value. Analysis of the crossover strain and amplitude as a function of the controlling parameters demonstrates that the linear theory is only applicable for a small range of amplitudes and strains. This renders unreliable the large strain prediction of wavelength selection based on the linear theory, especially for folding at high competence contrasts. To resolve this problem, the new finite amplitude theory is used to calculate the evolution of the growth rate spectra during progressive folding. The growth rate spectra exhibited splitting of a single maximum (predicted by the linear theory) into two maxima at large strains. This bifurcation occurred for both deformation modes. In contrast, the spectra of the cumulative amplification ratio (current over initial amplitude) maintained a single maximum value throughout. The wavelength selectivity is found to decrease at large strains, which helps explain the aperiodic forms of folds commonly observed in nature and the absence of long dominant wavelengths for high competence contrast folding. Calculation of the cumulative amplification spectra for different initial amplitude distributions, ranging from white to red noise, showed that the initial noise has a strong influence on the amplitude spectra for small strains. For larger strains, however, the cumulative amplification spectra were similar despite the strong difference in the initial noise. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

The initial stages of folding instabilities have been intensively investigated in recent decades (e.g., [1–3]). The main result of these linear theories is the identification of an amplification rate maximum at a certain fold wavelength, which is known as the dominant wavelength, and is expected to develop rapidly out of initially random irregularities (e.g., [4,5]). There are two particularly simple and frequently used solutions for folding of an elastic layer (referred to here as ‘elastic’ mode) and a viscous layer (referred to as ‘viscous’ mode) within a viscous matrix (e.g., [6]).

However, the linear theories have a number of shortcomings. To begin with, the deformation mode (viscous versus elastic) has to be known beforehand in order to apply the linear theory to natural folds. The viscoelastic theory, combining both deformation modes, resolves this problem by introducing a parameter $R$, the ratio of dominant viscous to dominant elastic wavelength, which determines if the competent layer is folded in a viscous ($R < 1$) or elastic ($R > 1$) mode [7].

Moreover, the linear theories predict dominant wavelengths significantly exceeding those observed in nature for viscosity contrasts in excess of 500 (viscous mode) or for ratios of layer-parallel stress ($P_0$) to a layer’s shear modulus ($G$) below 0.01 (elastic mode) (e.g., [8,9]). On the other hand, viscosity contrasts below 10 would result in a negligibly small growth rate of the instability compared to the background deformation (in the viscous mode). In the elastic mode, $P_0/G$ ratios must be less than 0.1, which is the maximum strength of any material. Therefore, the applicability of the linear theories is restricted to a very narrow range of controlling parameters. Some improvement has been achieved by using power law [10,11], elastoplastic [12], Mohr Coulomb [13] and viscoelastic rheologies [7]. However, uncommonly high power law exponents (> 10) or high $P_0/G$ ratios (> 0.01) are required to keep the small dominant wavelength predictions compatible with the large effective viscosity contrasts (e.g., up to $10^5$) and low viscosity of the matrix (e.g., below $10^{15}$ Pas for partially molten embedding media), presumably common in natural settings.

In addition, it is generally accepted that linear theories are restricted to small strains or amplitudes. Indeed, analogue experiments (e.g., [14–16]), nonlinear, semi-analytical theories [9] and numerical models (e.g., [7,17–19]) have shown a deceleration of the amplification after a certain amount of strain. However, none of these approaches provides: (i) a simple estimate for the strain or amplitude of the linear theory breakdown, (ii) identification of the breakdown mechanism, or (iii) a simple large strain solution for fold amplitude versus strain to replace the exponential expression obtained by the linear theory.

Finally, the linear theories predict sinusoidal fold shapes, whereas folds observed in nature often show localized and aperiodic forms (e.g., [20]). A linear thin-plate formulation equipped with somewhat arbitrary, nonlinear softening terms that emulates the response of the medium embedding the plate partly resolves this problem (e.g., [21,22]). Similarly, direct 2D numerical simulations (e.g., [7,19,23]) are able to generate the complex fold forms, but they do not identify the mechanism of their formation.

The shortcomings mentioned above demonstrate the necessity for a folding theory valid for finite amplitudes or large finite strains. However, finite amplitude (FA) folding is a nonlinear instability problem and, therefore, difficult to treat in a fully analytical way. The nonlinear buckling problem due to large deflections (often called post-buckling) is well known in the engineering literature, but mostly perfectly elastic beams or plates are considered, which are not embedded in a viscous matrix (e.g., [24,25]). A combined application of existing linear theories, asymptotic (or perturbation) methods (e.g., [26–28]) and numerical methods (e.g., [29,30]) provide a promising approach to solve the FA folding problem.

In this study, we use asymptotic and numerical methods in combination with the existing linear theories to derive a new FA theory of folding. The new FA theory resolves most of the shortcomings mentioned above. The theory is based on the thin-plate approximation for a competent layer embedded in an infinite, viscous matrix. The layer-parallel stress state of the competent layer is characterized by membrane (layer-aver-
aged) and fiber (linearly varying across the layer) stresses (Fig. 1) (e.g., [6,31]). The nonlinearity is introduced naturally through a weakening of the membrane stresses, which is caused by stretching of the competent layer during folding (Fig. 2). We consider two folding modes of the competent layer: (i) both membrane and fiber stresses are relaxed throughout the folding (the ‘viscous’ mode); and (ii) membrane stresses are relaxed whereas the fiber stresses are elastic (the ‘visco-elastic’ mode) (Figs. 1 and 2). These two cases cover a wide range of natural situations. The rate of fiber stress change during folding is much higher than the rate for the membrane stress and elasticity is therefore likely to be more important for the fiber stresses than for the membrane stresses. Formally, the viscous rheology for membrane stresses is applicable for folding of a viscoelastic layer if the ratio of the Maxwell relaxation time to the background strain rate (the Deborah number, $De$) is small (e.g., [7]).

2. Analytical theory for finite amplitude folding

The linear stage of folding is characterized by wavelengths that grow proportionally to their amplitude, i.e.:

$$\frac{\partial A(t)}{\partial t} = (1 + \alpha(t)) \dot{\varepsilon} A(t)$$

(1)

where $A(t)$, $t$, $\alpha(t)$ and $\dot{\varepsilon}$ are the amplitude, the
time, a dimensionless growth rate and the absolute value of the background strain rate for pure shear shortening, respectively (see Table 1 for explanation of the symbols used). According to our sign convention, compressive strain rates, strains and stresses are positive. For viscous and elastic competent layers, folded within a viscous matrix, the growth rate can be expressed as:

$$\alpha(t) = \alpha_0 P(\varepsilon)/P_0$$

(2)

where $P(\varepsilon)$ is the membrane stress averaged over layer thickness, $\varepsilon$ is the logarithmic strain, $P_0$ is the initial compressive membrane stress (equal to the initial layer-parallel stress, Fig. 1) and $\alpha_0$ is the initial, dimensionless growth rate given by:

$$\alpha_0 = \begin{cases} 
\frac{1}{2\pi} \left( \frac{\mu_m \mu_1}{H_0 \mu_0} + \frac{\pi^2 (H_0)}{3 \lambda_0} \right) \text{viscous layer} \\
\frac{\mu_m}{2} \frac{P_0}{\lambda_0} \left( 1 - \frac{4\pi^2 (H_0)}{3 \lambda_0} \frac{G}{P_0} \right) \text{elastic layer} 
\end{cases}$$

(3)

where $H_0$ is the initial layer thickness, $\mu_m$ is the viscosity of the matrix, $\mu_1$ and $G$ are the viscosity and the shear modulus of the layer, respectively, and $\lambda_0$ is the initial wavelength (e.g., [4,6]). $P_0$ is equal to $4\mu_1\varepsilon$ (cf., [6]) for the viscous layer and is a free parameter for the elastic layer. In this

Table 1
Symbols and their meaning

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>$\alpha_0$</td>
<td>constant, initial growth rate</td>
</tr>
<tr>
<td>$\alpha(t)$</td>
<td>current growth rate</td>
</tr>
<tr>
<td>$A_0$</td>
<td>initial fold amplitude</td>
</tr>
<tr>
<td>$\dot{A}_0$</td>
<td>initial fold amplitude divided over initial wavelength</td>
</tr>
<tr>
<td>$A(t)$, $A$</td>
<td>current fold amplitude</td>
</tr>
<tr>
<td>$\dot{A}(\varepsilon)$</td>
<td>current fold amplitude divided over current wavelength</td>
</tr>
<tr>
<td>$A_C$</td>
<td>crossover amplitude</td>
</tr>
<tr>
<td>$\dot{A}_C$</td>
<td>crossover amplitude divided over current wavelength</td>
</tr>
<tr>
<td>$D_e$</td>
<td>Deborah number</td>
</tr>
<tr>
<td>$\dot{\varepsilon}$</td>
<td>pure shear strain rate</td>
</tr>
<tr>
<td>$\dot{\varepsilon}_{arc}$</td>
<td>strain rate of arc length change</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>logarithmic strain</td>
</tr>
<tr>
<td>$\varepsilon_C$</td>
<td>crossover strain</td>
</tr>
<tr>
<td>$F_A$</td>
<td>finite amplitude</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus of the folded layer</td>
</tr>
<tr>
<td>$H_0$</td>
<td>initial layer thickness</td>
</tr>
<tr>
<td>$\Delta \theta_{u}$, $d_0$</td>
<td>initial, undeformed surface element of the layer</td>
</tr>
<tr>
<td>$\Delta \theta$, $d$</td>
<td>current, deformed surface element of the layer</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>initial fold wavelength</td>
</tr>
<tr>
<td>$\lambda_0$</td>
<td>initial dominant wavelength divided over initial thickness</td>
</tr>
<tr>
<td>$\lambda$, $\dot{\lambda}(t)$, $\dot{\lambda}(\varepsilon)$</td>
<td>current fold wavelength</td>
</tr>
<tr>
<td>$\dot{\lambda}(t)$</td>
<td>current fold wavelength divided by current thickness</td>
</tr>
<tr>
<td>$L_{arc}$</td>
<td>current fold arc length</td>
</tr>
<tr>
<td>$L_{arc0}$</td>
<td>initial fold arc length divided over initial wavelength</td>
</tr>
<tr>
<td>$\dot{L}_{arc}$</td>
<td>current fold arc length divided over current wavelength</td>
</tr>
<tr>
<td>$\mu_l$</td>
<td>viscosity of the folded layer</td>
</tr>
<tr>
<td>$\mu_m$</td>
<td>viscosity of the matrix</td>
</tr>
<tr>
<td>$N_L$</td>
<td>nonlinear term in finite amplitude solution</td>
</tr>
<tr>
<td>$O(\varepsilon)$</td>
<td>order symbol</td>
</tr>
<tr>
<td>$P_0$</td>
<td>initial membrane stress</td>
</tr>
<tr>
<td>$\sigma_{xx}$</td>
<td>current membrane stress</td>
</tr>
<tr>
<td>$R$</td>
<td>ratio of dominant viscous to dominant elastic wavelength</td>
</tr>
<tr>
<td>$\sigma_{xx}$</td>
<td>layer-parallel stress</td>
</tr>
<tr>
<td>$\sigma_{ss}$</td>
<td>fiber stress</td>
</tr>
<tr>
<td>$S(\varepsilon)$</td>
<td>shape function, used in Taylor series for fold arc length</td>
</tr>
<tr>
<td>$\tau_{top}$</td>
<td>shear stress imposed by matrix onto top of layer surface</td>
</tr>
<tr>
<td>$\tau_{bot}$</td>
<td>shear stress imposed by matrix onto bottom of layer surface</td>
</tr>
<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$\Delta \eta$</td>
<td>velocity difference along the folded layer</td>
</tr>
</tbody>
</table>
study, the membrane stresses are considered to be viscous, thus \( P_0 = 4\mu\dot{\varepsilon} \) for both our ‘viscous’ and ‘viscoelastic’ modes. This implies that the Deborah number is small:

\[
\text{De} = \frac{W_l}{\nu} = \frac{P_0}{4\mu}.
\]

In the linear theory, fold amplification is controlled by the dominant wavelength, which is presumably selected and ‘locked’ at the initial stage of folding. For an incompressible material (where Poisson’s ratio is equal to 0.5), the dominant wavelength can be expressed as (e.g., [6]):

\[
\hat{\lambda}_0 = \begin{cases} 
2\pi \sqrt{\frac{\mu_1}{6\mu_m}} & \text{viscous layer,} \\
2\pi \sqrt{\frac{G}{P_0}} = 2\pi \sqrt{\frac{\mu_1}{6\mu_m} \frac{1}{R}} & \text{elastic layer, applicable for } R > 1
\end{cases}
\]

where \( \hat{\lambda}_0 \) is the dominant wavelength normalized over the initial layer thickness and \( R = 6^{1/3}(\mu_1/\mu_m)^{1/3}(P_0/G)^{1/2} \) is the ratio of dominant viscous to dominant elastic wavelength, which is a parameter controlling the choice of rheological response [7]. The growth rate of the dominant

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**Fig. 3.** Large strain dynamics and kinematics of the thin layer embedded in a weak matrix. (A) Force balance along the folded layer. The variation of the membrane force along a small increment of the layer is balanced by the shear stresses that are imposed by the embedding matrix onto the top (\( \tau_{\text{top}} \)) and bottom (\( \tau_{\text{bot}} \)) layer segments \( \Delta l \). Negligible shear stresses within the low viscosity, embedding matrix lead to effectively constant membrane stress if a constant layer thickness (thin-plate) is assumed. (B) Local layer-parallel pure shear kinematics are assumed to be applicable even after large fold limb rotations due to negligible shear deformation within the layer. (C) Sketch of mapping from undeformed to deformed state, explaining the calculations of the cumulative arc length rate using Eq. 6.
wavelength is obtained by substituting Eq. 4 into Eq. 3:

\[
\alpha_0 = \begin{cases} 
\left(\frac{\hat{\lambda}_0}{\pi}\right)^2 = \left(\frac{4}{3} \frac{\mu_1}{\mu_m}\right)^{2/3} & \text{viscous layer, applicable for } R < 1 \\
\frac{\pi}{3\hat{\lambda}_0} \frac{P_0}{\mu_m \hat{\varepsilon}} = \left(\frac{4}{3} \frac{\mu_1}{\mu_m}\right)^{2/3} R & \text{elastic layer, applicable for } R > 1
\end{cases}
\] (5)

The reason that the ratio of dominant viscous to dominant elastic wavelength, \( R \), controls the choice of deformation mode (as demonstrated by [7]) is clear from Eq. 5, since the growth rate for the viscous folding mode is faster than the elastic mode if \( R < 1 \), and vice versa.

Assuming a constant strain rate and constant membrane stress, \( P(\varepsilon) = P_0 \), the integration of Eq. 1 yields the exponential solution \( A(t) = A_0 \exp((1+\alpha_0)\hat{\varepsilon}t) \) for the initial stage of folding, where \( A_0 \) is the initial amplitude (e.g., [4,6,9]). However, the membrane stress is inevitably altered at later stages of folding (e.g., [32]). This is because the assumed sinusoidal shape of the fold and its exponentially growing amplitude implies a fast increase in the arc length \( L_{arc} \) of the competent layer, which in turn causes a weakening of the compressive membrane stress due to the strong resistance of the competent layer to stretching.

The FA approximation is made by assuming that the linear theory, based on an initial dominant wavelength \( \hat{\lambda}(t) = \hat{\lambda}_0 \), is valid incrementally (Eqs. 1–5) throughout the FA folding. Under pure shear, the change in the wavelength with time is defined as \( \partial\hat{\lambda}(\varepsilon)/\partial t = -\hat{\lambda}(t)\dot{\varepsilon} \). In the following derivations, the chain rule of differentiation, \( \partial\hat{\lambda}(t)/\partial t = (\partial\hat{\lambda}(\varepsilon)/\partial \varepsilon)(\partial \varepsilon/\partial t) = (\partial\hat{\lambda}(\varepsilon)/\partial \varepsilon)\dot{\varepsilon} \), is used to convert the time to strain derivatives, which allows the theory to be independent of strain rate variation through time. Therefore, the wavelength changes with strain according to \( \partial\hat{\lambda}(\varepsilon)/\partial \varepsilon = -\hat{\lambda}(t)\dot{\varepsilon} \), the logarithmic strain being \( \varepsilon = \ln(\hat{\lambda}_0/\hat{\lambda}(\varepsilon)) \).

According to the force balance along the layer, the variation of the membrane force (membrane stress times the layer thickness) along the layer must be balanced by the shear stresses imposed by the embedding matrix onto the layer’s surface (cf., [33,34], Fig. 3A). It is assumed here that the membrane stress is constant along the layer, because the shear stresses in the low viscosity matrix are small compared to the stresses within the layer. In accordance with viscous rheology, the membrane stress must be proportional to the ‘local’ layer-parallel (and layer-averaged) strain rate, \( \dot{\varepsilon}_{arc} = -(\partial \Delta l/\partial t)/\Delta l \) (Fig. 3B,C), which must, therefore, also be constant along the layer. The cumulative change of the arc length is given by:

\[
\frac{\partial}{\partial t}L_{arc} = \frac{\partial}{\partial t}\left(\int_0^{L_{arc}} dt \right) = \frac{\partial}{\partial t}\left(\int_0^{L_{arc}} \frac{\partial l}{\partial l_0} dl_0 \right) = \\
\int_0^{L_{arc}} \frac{\partial^2 l}{\partial l_0^2} \frac{\partial l}{\partial l_0} dl_0 = -\int_0^{L_{arc}} \frac{\partial}{\partial l_0} (\dot{\varepsilon}_{arc}) dl_0 = \\
-\dot{\varepsilon}_{arc} \int_0^{L_{arc}} \frac{\partial l}{\partial l_0} dl_0 = -\dot{\varepsilon}_{arc} \int_0^{L_{arc}} dl = -\dot{\varepsilon}_{arc} L_{arc}
\]

Therefore, the arc length (membrane) strain rate can be expressed as:

\[
\frac{\partial}{\partial t}L_{arc} = -\dot{\varepsilon}_{arc} L_{arc}
\]
\[
\dot{\varepsilon}_{\text{arc}} = -\left( \frac{\partial L_{\text{arc}}}{\partial t} \right) / L_{\text{arc}} = -\dot{\varepsilon} \left( \frac{\partial L_{\text{arc}}}{\partial \varepsilon} \right) / L_{\text{arc}}
\]  
(7)

and the viscous membrane stress, normalized over its initial value \(4\mu \dot{\varepsilon}\), can be written as:

\[
\frac{P(\varepsilon)}{P_0} = -4\mu \dot{\varepsilon} \left( \frac{1}{L_{\text{arc}}} \frac{\partial L_{\text{arc}}}{\partial \varepsilon} - 1 \right) \frac{1}{4\mu \dot{\varepsilon}} = 1 - \frac{1}{L_{\text{arc}}} \frac{\partial L_{\text{arc}}}{\partial \varepsilon}
\]  
(8)

where \(\dot{L}_{\text{arc}} = L_{\text{arc}}/\dot{\lambda}\) is the arc length normalized over the current wavelength. An expression for the normalized arc length versus fold amplitude can be written in the general form:

\[L_{\text{arc}} = 1 + S(\varepsilon) \dot{A}(\varepsilon)^2\]  
(9)

Here \(\dot{A}(\varepsilon) = A(\varepsilon)/\dot{\lambda}(\varepsilon)\) is the current fold amplitude normalized over the current wavelength and \(S(\varepsilon)\) is a function to account for fold shape. For sinusoidal fold shapes, the constant value:

\[S(\varepsilon) = \pi^2\]  
(10)

is derived by keeping only the first two terms of the Taylor expansion of \(\dot{L}_{\text{arc}}\) for small amplitudes:

\[
\dot{L}_{\text{arc}} = \frac{1}{\dot{\lambda}} \int_0^{\dot{\lambda}} \sqrt{1 + \left( A(\varepsilon) \frac{d}{dx} \sin \left( \frac{2\pi x}{\lambda} \right) \right)^2} \, dx \approx \\
1 + \pi^2 \dot{A}(\varepsilon)^2
\]  
(11)

For large amplitudes the shape of folds does not stay sinusoidal and, therefore, the range of applicability of the Taylor expansion of \(\dot{L}_{\text{arc}}\) (only valid for small \(A/\lambda\)) and the \(\dot{L}_{\text{arc}}\) versus \(A/\lambda\) relationship outside this range is determined by numerical experiments. Three numerical experiments were performed for different rheological parameter sets representing three different deformation modes (Fig. 4): (i) pure viscous folding \((R = 0)\); (ii) viscoelastic folding with quasi-viscous behavior \((R = 0.7)\); and (iii) viscoelastic folding with quasi-elastic behavior \((R = 2)\). In Fig. 5A the curve for \(\dot{L}_{\text{arc}}\) with \(S(\varepsilon) = \pi^2\) and the numerically calculated arc length to wavelength ratios are plotted versus \(A/\lambda\). Despite this wide range of rheological behavior, all three numerical fold shapes follow the same curve \(\dot{L}_{\text{arc}} = 1 + \pi^2 \dot{A}(\varepsilon)^2\) (derived Taylor expansion assuming small amplitude to wavelength ratios and sinusoidal fold shape) up to the surprisingly high value of
$A/\lambda \approx 0.2$, which corresponds to a dip angle at the inflection point of about 50° ($= \arctan(2\pi A/\lambda)180/\pi$). Furthermore, all three numerical fold shapes follow, again independently of the rheology, a general curve:

$$S(\varepsilon) = \pi^2/(1 + 3A(\varepsilon)^2) \quad (12)$$

which was found to approximate the fold shape evolution best for $A/\lambda > 0.2$ and to comply with the curve found by the Taylor expansion for $A/\lambda < 0.2$ (see also Fig. 5B). Most results presented in this paper are for $A/\lambda < 0.2$ and, therefore, are independent of the numerically fitted general curve (Eq. 12).

Finally, the FA equation for the fold amplitude normalized over the current wavelength is obtained by substituting Eqs. 2 and 8 into Eq. 1 and solving the resulting equation for $\partial A(\varepsilon)/\partial \varepsilon$. After some rearrangement of terms the nonlinear FA equation, valid for the two modes of rheological response of the competent layer, is given as:

$$\frac{\partial A(\varepsilon)}{\partial \varepsilon} = (2 + \alpha_0) \frac{A(\varepsilon)}{1 + N_L} \quad (13)$$

Note that the summand 2 appears in this equation because the amplitude is normalized over the current wavelength and the background amplification for this ratio is twice as fast as for the amplitude.
plitude alone. The nonlinear term, $N_L$, has the form:

$$N_L = \alpha_0 \frac{\hat{A}(\epsilon)}{L_{arc}} \frac{\partial L_{arc}}{\partial \hat{A}(\epsilon)}$$

(14)

Substituting Eqs. 9 and 12 into Eq. 14 yields:

$$N_L = \alpha_0 \frac{2\pi^2 \hat{A}(\epsilon)^2}{(3\pi^2 + 9) \hat{A}(\epsilon)^4 + (\pi^2 + 6) \hat{A}(\epsilon)^2 + 1} \approx 2\alpha_0 \pi^2 \hat{A}(\epsilon)^2 + O(\hat{A}(\epsilon)^4)$$

(15)

Eq. 13 has an exponential solution (valid for small strains) if the nonlinear term is neglected. An estimate of the crossover amplitude, $\hat{A}_C$, for viscous and viscoelastic modes is derived by assuming that the exponential solution breaks down when the nonlinear term is of the order of unity. The approximated nonlinear term in Eq. 15 yields the crossover amplitude:

$$\hat{A}_C \approx 1/(\pi \sqrt{2\alpha_0})$$

(16)

Matching $\hat{A}_C$ to the exponential solution provides an expression for the crossover strain:

$$\gamma_C = \frac{1}{2 + \alpha_0} \ln \left( \frac{\hat{A}_C}{\hat{A}_0} \right)$$

(17)

An analytical solution of Eq. 13 using the exact nonlinear term in Eq. 14 can be written in implicit form as:

$$\epsilon = \ln \left( \frac{\hat{L}}{\hat{A}} \right) = \ln \left( \frac{L_{arc}}{L_{arc0}} \right)^{\alpha_0/(2 + \alpha_0)} + \ln \left( \frac{\hat{A}(\epsilon)}{\hat{A}(\epsilon)_0} \right)^{1/(2 + \alpha_0)}$$

(18)

where $\hat{L}_{arc0}$ is the initial arc length to wavelength ratio. The above solution is valid for an arbitrary relationship of arc length versus amplitude and is, therefore, unaffected by the choice of the shape factor (Eqs. 10 and 12). This solution can be transformed to another form as:

$$\hat{A}(\epsilon) = \hat{A}_0 \exp \left( (1 + \alpha_0)\epsilon \left( \frac{L_{arc0}}{L_{arc}} \right)^{\alpha_0} \right)$$

(19)

This form shows that the main difference between the FA solution and the linear solution is the strong power law dependence ($\alpha_0 \gg 1$) of the change in the fold arc length.

It is important to note here that the FA solution is a general solution of a dynamic problem. In order to predict the fold amplification versus strain, the theory requires an additional kinematic constraint for arc length versus amplitude, which is a function of the fold shape and potentially strongly controlled by rheology (e.g., [17]). How-

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**Fig. 8.** The crossover strains at which the linear theory breaks down. (A) As viscosity contrasts are increased, the crossover strains decrease strongly for fixed $\hat{A}_0/\lambda_0$, which means that the linear theory is valid only for the first few percent of strain. (B) The crossover amplitude for viscous and viscoelastic layers versus the parameter $R$ (double logarithmic plot) for the small strain approximation exp$(\gamma_C) \approx 1$. For $R < 1$ (viscous mode) the crossover amplitude is constant. It starts to decrease for $R > 1$ (viscoelastic mode).
ever, the important features of the FA solution, such as the deviation from exponential growth and the transition into layer length controlled growth, are derived by using the analytical Taylor expansion of the arc length and assuming sinusoidal fold shape. We demonstrate that these assumptions (i.e., $S(\varepsilon) = \pi^2/3$) are valid up to $A/\lambda < 0.2$ corresponding to a maximum limb dip of $50^\circ$. For the larger amplitudes, the numerically derived function $S(\varepsilon) = \pi^2/(1+3A(\varepsilon)^2)$ can be used as a better kinematic constraint on the arc length evolution to increase the accuracy of the FA theory predictions.

3. Verification of the FA theory and derivation of FA spectra

In this section, the FA solution (Eq. 18) is compared with the exponential solution predicted by the linear theory and numerical solutions of the complete 2D folding problem obtained with the combined spectral/finite-difference method [7] (Figs. 6 and 7). The amplification ratio given by the FA solutions is very close to that obtained from numerical simulations for very large strains, providing a confirmation that the simplifications made in the FA theory are reasonable (Fig. 6). Furthermore, the FA solution starts to deviate from the exponential solution at around the same value for strain and $A/H_0$ as the numerical solution (Fig. 7). Therefore, the crossover amplitude and strain, predicted on the basis of the FA solution, give reliable limits on the linear theory.

The crossover strain at which the linear solution starts to deviate from the FA solution is presented in Fig. 8A. As expected, higher values of $A_0/\lambda_0$ (i.e., higher initial limb dip) cause earlier breakdown of the exponential assumption for fixed rheological parameters (cf. Eq. 17 and Fig. 8A). The crossover amplitude in Eq. 16 is normalized over the current wavelength. Multiplying both sides of Eq. 16 by $\lambda_0$ results in an expression for $A_C/H_0\exp(\varepsilon_C)$ that only depends on $R$. For small crossover strains (see Fig. 8A), the value of $\exp(\varepsilon_C)$ is approximately unity. With this assumption, $A_C/H_0$ can be plotted versus the single parameter $R$ (Fig. 8B) and has the constant value $A_C/H_0 \approx 1.65$.

Fig. 9. Evolution of growth rate spectra with progressive shortening (numbers in legend) for three sets of rheological parameters. The graphs are for ‘white’ noise (initial amplitudes are the same). For small shortening the growth rate spectra are equal to those predicted by the linear theory. With progressive shortening, the growth rate spectra yield two maximal growth rates, corresponding to wavelengths smaller and larger than the dominant one. Note the smaller shortening range for $R = 2$. 
1/\sqrt{2}$ for $R < 1$ (the viscous mode) and is proportional to $1/\sqrt{2 R^{3/2}}$ for $R > 1$ (the viscoelastic mode).

The FA solution provides normalized growth rates, $\varepsilon(t) = (\varepsilon(t)/\varepsilon(0)/A(t) - 1$, for different initial wavelength to thickness ratios, to give the growth rate spectra. In contrast to the linear theory, these FA spectra are dependent on strain and initial amplitude and remain valid for amplitudes larger than the crossover one. The change of the growth rate spectra with progressive shortening is presented in Fig. 9 for three sets of rheological parameters. The maximum corresponding to wavelengths larger than the initial dominant one rapidly shifts towards larger wavelengths.

In order to evaluate the effect of splitting of the growth rate maximum on dominant wavelength selection, the spectra of $A/H_0$ are calculated for the three different sets of rheological parameters and for different amounts of shortening (Fig. 10). The difference between the spectra for growth rate and current amplitude is that the growth rate represents the incremental amplification whereas the current amplitude represents the total amplification accumulated since the onset of folding. For all three settings, the dominant wavelength stays practically unaltered with progressive shortening. In other words, the selectivity of the dominant wavelength becomes weaker with progressive shortening, especially for folding at high competence contrasts and folding in viscoelastic modes. Similar results were obtained by Mancktelow [35] using finite element simulations.

Furthermore, the FA theory can be used to calculate contours of growth rate, amplification ratio, $A(t)/A_0$, and the ratio $P(t)/P_0$, during progressive shortening within the two coordinates $\lambda_0/H_0$ and $\lambda_0/\lambda$ (Fig. 11). The contours of $P(t)/P_0$ indicate that the minimum layer-parallel stress (which is proportional to the minimum force in the linear stage, cf. [5]) corresponds to this dominant wavelength.

All the above spectra were calculated for a distribution of initial amplitudes corresponding to uncorrelated ‘white noise’ (i.e., all initial amplitudes are the same). However, the FA spectra inevitably depend on the distribution of initial amplitudes due to the nonlinearity of the FA theory. Therefore, a sensitivity study for the fold amplification spectrum was carried out for three different distributions of initial amplitudes corresponding to initial uncorrelated ‘white noise’ (all initial amplitudes are the same and scale as $V_0^0$) and two long-wavelength dominated ‘red noises’, where the initial amplitudes scale as $V_0^1$ and $V_0^2$ (e.g., [36]) (Fig. 12). For small shortening (1.02), the amplification is different for the three initial wavelength scaling factors $n=0$, 1, 2 where $n$ is the exponent in $\lambda_0^n$. For $n=0$ and 1, the dominant wavelength is already developed, whereas for $n=2$
the initial amplitude distribution is still observable. After a shortening of 1.11 a maximum starts to develop also for \( n = 2 \), and becomes fully developed at a shortening of 1.28. At a shortening of 1.65 the three amplitude spectra develop the same dominant wavelength and are similar, although the initial amplitude spectra were very different.

4. Discussion

The calculation of the crossover strains (Fig. 8A) shows that for high viscosity contrast or high \( P_0/G \) the exponential solution breaks down after only a small amount of shortening. As a consequence, for folding having these parameter sets, the growth rate spectra calculated from the linear theory are no longer applicable and the development of a dominant wavelength must be questioned. The growth rate and \( A/H_0 \) spectra predicted by the FA theory (Figs. 9 and 10) may help to explain two common observations on natural folds. First, a number of natural fold trains show variations in wavelength although the layer thickness is more or less constant (e.g., [21]). This can be explained by the development of flat and non-selective \( A/H_0 \) spectra at large strains (Fig. 10). Second, wavelength to thickness ratios observed in nature are usually below 30 (e.g., [8,37]), implying that viscosity contrasts in nature are never larger than around 500. This contradicts
experimental observations of power law creep (e.g., [38]), which suggest that effective viscosity ratios in natural settings may exceed several orders of magnitude. The apparent contradiction can be explained by the FA theory. It predicts that folds with high viscosity contrast have only a few percent of strain available to develop the dominant wavelength, especially when their initial amplitude to wavelength ratio is larger than around 0.01 (Fig. 8). Moreover, the FA theory predicts that the incremental growth rate spectra exhibit very sharp maximal values at wavelengths smaller than the dominant one (Fig. 9).

Comparison of folding spectra for three different initial noises (the power law exponents ranging from 0 to 2) demonstrated a weak sensitivity at the FA stages. The ‘white and red noise’ settings used in this sensitivity analysis represent a wide range of natural surfaces. The white noise (the amplitude scales as $\lambda_0^n = $constant) is an extremely rough interface having zero correlation length. Red noise (the amplitude scales as $\lambda_0^n$, where $n$ is a constant) is characterized by strong near-neighbor correlation and increasing degree of smoothness with growing power law exponent $n$. Most natural surfaces are considered to lie within a range of exponents $n$ between 0 and 1 (e.g., [36]). As presented in Fig. 12, the $A/\lambda$ spectra developed are very similar for the ‘end member noises’ corresponding to $n=0$ and $n=1$. Moreover, the slope of $A/\lambda$ for increasing wavelengths, larger than the dominant one, becomes flatter with progressive shortening. This reflects the weak selectivity already shown in Fig. 10 for the $A/H_0$ spectra. For $n=2$, the initial amplitude distribution is preserved for shortening more than 1.1. However, after a shortening of 1.65 the distribution of $A/\lambda$ is more or less the same for all three initial ‘noises’. This suggests that the type of initial ‘noise’ has only a minor influence on fold development. Indeed, Mancktelow [35] showed, using numerical (finite element method) simulations, that the initial probability distribution and the fractal dimension of these distributions have only a minor influence on the wavelength of folds developed at the finite amplitude at which they can be studied.

5. Summary and conclusions

A simple nonlinear FA theory has been derived that accounts for membrane stresses during progressive folding. It is argued that the weakening of the compressive membrane force causes the breakdown of the initially exponential fold amplification and its sharp transition into a much slower, layer length controlled growth. The FA theory
shows that for high viscosity contrasts (strains at which the linear theory breaks down) or values of numerical folds at large amplitudes and is given by a simple relationship of the fold wavelength, which is a parameter controlling the choice of rheological response during folding [7]. Furthermore, a large strain solution predicting the further evolution of the fold amplitude is derived, which is:

\[ A(\varepsilon) = A_0 \exp\left(1 + \alpha_0\right) e^{\left(\frac{L_{arc0}}{L_{arc}}\right)^\alpha_0} \]

where \( L_{arc} \) is the fold arc length normalized over the fold wavelength, \( \varepsilon = \ln(\lambda_0/\lambda) \) is the logarithmic strain, and \( A_0, \lambda_0 \) and \( L_{arc0} \) are initial amplitude, wavelength and the normalized arc length, respectively. \( L_{arc} \) is obtained as a function of the fold amplitude and wavelength by Taylor expansion at small amplitudes and by fitting the shapes of numerical folds at large amplitudes and is given by a simple relationship \( L_{arc} = 1 + \pi^2 A(\varepsilon)^2 / (\lambda^2 + 3A(\varepsilon)^2) \) valid up to threefold shortening.

The calculation of the crossover amplitude and strains at which the linear theory breaks down shows that for high viscosity contrasts (\( > 1000 \)) or values \( R > 1 \) the linear theories are only valid for a small amount of strain (Fig. 8). The evolution of the incremental growth rate spectra during progressive finite amplitude folding can also be calculated (Fig. 9). The incremental spectrum shows that the single amplification rate maximum, which occurs at small strain, bifurcates into two maxima located at longer and shorter wavelengths. The offset between the two maxima increases with increasing strain. As a result of this strain dependence of the incremental amplification rate, the spectrum of \( A/H_0 \) exhibits a continuous range of wavelengths characterized by nearly the same finite amplification as the dominant one in the case of high viscosity contrasts or \( R > 1 \) (Fig. 10). It may be speculated that the weaker selectivity of the finite strain spectrum is responsible for the aperiodic shapes of natural folds and for the absence of long dominant wavelengths for high competence contrast folding. The spectra of the stress ratio \( (P/P_0) \) confirm that the dominant wavelength is the one corresponding to the minimum layer-parallel stress for a given strain (Fig. 11). The influence of different initial noise decreases with increasing strain resulting in more or less the same amplitude spectra for large strains (Fig. 12). In conclusion, the results obtained in this study advocate the combined application of analytical, asymptotic and numerical methods to provide an insight into nonlinear problems in geodynamics.

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