Abstract

The presented equation describes amplitude growth during viscous single-layer folding (buckling) up to high amplitudes. The equation relates the dimensionless fold amplitude (i.e. ratio of amplitude to wavelength) to the stretch (ratio of initial wavelength to instantaneous wavelength) for given values of the viscosity contrast between layer and surrounding material and the initial ratio of amplitude to wavelength. The amplification equation is suitably scaled so that all amplitude versus stretch curves for different values of viscosity contrasts and initial amplitudes fall onto essentially a single curve. The scaled amplification equation allows for representing fold amplification of viscous single-layers by a singular curve. The scaling parameter is the crossover strain, which is an estimate for the amount of strain that is accumulated during the initial stages of folding where the amplitude grows exponentially with strain. The singular curve allows quantifying the universal boundaries between the three folding stages, namely nucleation, amplification and kinematic growth. The scaled amplification equation is verified by numerical (finite element method) simulations of folding of single layers with initial random perturbations of the layer interfaces. The amplification equation describes the amplification of single folds within fold trains successfully, although the folds are neither regular nor periodic and vary considerably in shape. The easily measurable parameters, vertical and horizontal hinge distance, are shown to be good approximations for the analytical parameters amplitude and wavelength, respectively.

Keywords: Single-layer folding; Scaling; Finite elements; Analytical solution; Viscous flow

1. Introduction

Single-layer folding originates from a mechanical instability. This instability has been studied intensively by structural geologists because folding occurs frequently and on all scales during rock deformation (e.g., Johnson and Fletcher, 1994, and references therein). Furthermore, the results and interpretations derived from single-layer folding studies can be extended and applied to other geologically relevant mechanical instabilities, such as crustal and lithospheric folding, folding of isolated finite length layers, salt diapirism, pinch and swell structures or necking (e.g., Biot, 1961; Smith, 1977; Johnson and Fletcher, 1994; Schmalholz et al., 2002; Schmid et al., 2004).

Analytical results for folding have been mainly derived using linear stability analysis (e.g., Johnson and Fletcher, 1994), which usually assumes a sinusoidal fold shape characterized by a wavelength and amplitude (Fig. 1). The analytical results show that the growth rate of the fold amplitude is a function of the fold wavelength (i.e. dispersion) and that the growth rate versus wavelength curve exhibits a maximum (Biot, 1961; Fletcher,
The wavelength corresponding to the maximal growth rate is termed the dominant wavelength (Biot, 1961). In this study, only Newtonian viscous rheologies are considered. Therefore, the growth rate only depends on the wavelength and the viscosity contrast between the layer and the surrounding material (referred to here as matrix), if the growth rate and the wavelength are non-dimensionalised by the layer-parallel shortening strain rate and the layer thickness, respectively (Fletcher, 1977). Analogue and numerical experiments have confirmed the analytical results, which are strictly valid only for simple model set-ups and infinitesimal amplitudes (e.g., Ramberg, 1963; Hudleston, 1973; Schmalholz and Podladchikov, 1999). However, comparing analytical results with realistic fold shapes is problematic because the analytical parameters amplitude and wavelength have a strict mathematical definition, whereas measurable geometrical parameters describing similar properties of natural folds, such as vertical and horizontal hinge distances, do not have the same mathematical properties (Fig. 1). One aim of this study is to show that the horizontal and vertical hinge distances of natural folds are good approximations to the analytical parameters wavelength and amplitude, respectively, in the context of describing fold amplification.

Fold evolution has often been described with some kind of fold amplitude versus horizontal shortening curve (e.g., Hudleston, 1973) and has frequently been subdivided into different stages, such as for example nucleation or amplification (e.g., Cobbold, 1976; Ghosh, 1993). Such subdivisions have been made qualitatively without providing quantitative values for the boundaries between these folding stages. One reason is that the value of the fold amplitude essentially depends on the three parameters initial amplitude, growth rate and amount of shortening. Therefore, an infinite number of amplitude versus shortening curves is possible for different values of growth rates and initial amplitudes. This dependence on several parameters prohibits defining specific values quantifying horizontal shortening which subdivide the folding stages and are additionally valid for all possible amplitude versus shortening curves. In this study, a scaled amplification equation is presented which relates the fold amplitude to essentially a single parameter, termed the scaled stretch, and enables defining specific values of the scaled stretch which define the boundaries between the three folding stages nucleation, amplification and kinematic growth. These boundary values are valid for all viscous single-layer folds originating from different initial amplitudes and having different growth rates or, alternatively, viscosity contrasts. The study further shows that fold amplification becomes insensitive to the viscosity contrast during the second, amplification stage and that during kinematic growth the fold arclength stays nearly constant.

The presented analytical results are verified by numerical simulations of viscous single-layer folding. The applied self-developed numerical algorithm is based on the finite element method and described in the Appendix. The scaled amplification equation is further applied to a natural single-layer fold.

The two main aims of this study are to show first, that the vertical and horizontal hinge distance of folds are good approximations for the mathematical parameters amplitude and wavelength, respectively, and second, that the scaled amplification equation allows quantifying the universal boundaries between three folding stages of viscous single-layers.

### 2. Scaled amplification equation

The following derivations are based on the results of the linear stability analysis for viscous single-layer folding (Biot, 1961; Fletcher, 1977). Only the dominant wavelength and the corresponding maximal growth rate are considered for the following analysis. The general derivation of the applied finite amplitude solution is given in Schmalholz and Podladchikov (2000) and applications have been presented in Schmalholz and Podladchikov (2001). Only the basic mathematical steps are presented here in a form suitable for the purpose of scaling.

The classical setup for folding of a single viscous layer embedded in a less viscous matrix under pure shear shortening in two dimensions is considered (Fletcher, 1977; Johnson and Fletcher, 1994). The amplitude of a
fold with an initially sinusoidal shape can be calculated using linear stability analysis by

\[ A = A_0 \exp((1 + \alpha_0)e) \]  

(1)

where \( A, A_0, \alpha_0 \) and \( e \) are the amplitude, the initial amplitude, the dimensionless initial maximal growth rate and the absolute value of the pure shear strain, respectively (Fig. 1). The maximal growth rate depends on the viscosity contrast, \( R \), between layer and matrix and a thin-plate analysis provides \( \alpha_0 = (4R/3)^{2/3} \) \( (Biot, 1961; \text{Schmalholz and Podladchikov}, 2000) \). For pure shear, the shortening of the fold wavelength (here the dominant one) is controlled by \( e \) through the relationship \( (\text{Johnson and Fletcher}, 1994) \)

\[ \lambda = \lambda_0 \exp(-e) \]  

(2)

where \( \lambda \) and \( \lambda_0 \) are the instantaneous and initial fold wavelength, respectively (Fig. 1). Dividing Eq. (1) by Eq. (2) yields

\[ \frac{A}{\lambda} = \frac{A_0}{\lambda_0} \exp((2 + \alpha_0)e) \]  

(3)

where \( \bar{A} = A/\lambda \) and \( \bar{A}_0 = A_0/\lambda_0 \). The amplitude is here divided by the wavelength to make the results dimensionless and the dimensionless amplitude \( \bar{A} \) can be viewed as amplitude measured in units of the fold wavelength. Solving Eq. (3) for \( e \) yields

\[ e = \frac{1}{2 + \alpha_0} \ln \left( \frac{\bar{A}}{\bar{A}_0} \right) \]  

(4)

with \( \ln \) being the natural logarithm. Introducing the stretch \( S = \lambda_0/\lambda \) and using the relationship \( e = \ln (S) \), which follows from Eq. (2), yields \( (\text{Johnson and Fletcher}, 1994) \)

\[ S = \frac{\bar{A}}{\bar{A}_0} \]  

(5)

Eq. (5) relates the dimensionless fold amplitude to the stretch for given values of the initial amplitude and growth rate. It describes fold amplification accurately up to fold limb slopes of about 15°, depending on the viscosity contrast \( (\text{Chapple}, 1968; \text{Schmalholz and Podladchikov}, 2000) \). For considerably larger limb...
slopes, Eq. (5) is not suitable to describe fold amplification. Schmalholz and Podladchikov (2000) presented an amplification equation valid up to limb slopes of ca. $70^\circ$:

$$ S = \left( \frac{A}{A_0} \right)^{\frac{1}{2}} \left( \frac{L}{L_0} \right)^{\frac{2n}{2n-1}} $$

(6)

where the arclength, $L$, is expressed as

$$ L = \frac{L}{\lambda} = 1 + \frac{\pi^2 A^2}{1 + 3A^2}. $$

(7)

Eq. (6) shows that the finite amplitude solution consists of the classical solution presented in Eq. (5) multiplied by a correction term that takes into account the relative change of the fold arclength during amplification. Eq. (7) defining $L$ is derived in Schmalholz and Podladchikov (2000). The value for $L_0$ is determined by substituting $\lambda = \lambda_0$ and $A = A_0$ into Eq. (7). The value of $L$ is an approximation of the fold arclength based on the values of the amplitude and wavelength (Fig. 1). Schmalholz and Podladchikov (2000) also determined a crossover strain, which is an estimate for the amount of horizontal strain accumulated during the linear stages of folding where Eq. (5) provides an accurate amplitude versus stretch curve. The crossover strain is (Schmalholz and Podladchikov, 2000, 2001)

$$ e_C = \left( \frac{\pi \sqrt{2 \alpha_0 A_0}}{4A_0} \right)^{\frac{1}{2n-1}}. $$

(8)

Dividing both sides of Eq. (6) with $e_C$ yields

$$ S_S = \left( \frac{A}{A_0} \right)^{\frac{1}{2n-1}} \left( \frac{L}{L_0} \right)^{\frac{2n}{2n-1}} \left( \frac{\pi \sqrt{2 \alpha_0 A_0}}{4A_0} \right)^{\frac{1}{2n-1}} $$

(9)

where $S_S = S/e_C$ is termed the scaled stretch. The bulk shortening, $\varepsilon$, can be calculated using $\varepsilon = 1 - 1/S$. The maximum limb dip at the inflection points, $\theta$, can be calculated in degrees using the relationship $\theta = \arctan \left( \frac{2\pi A}{\lambda} \right) 180 / \pi$ assuming a sinusoidal fold shape.

![Scaled stretch (\(\lambda_0/\lambda, e_C\))](image)

Fig. 3. The general amplification curve of single-layer folds. The black line is calculated for $\alpha_0 = 15$ (corresponding to a viscosity contrast about 50) and $A_0/\lambda_0 = 3e^{-3}$ (corresponding to a limb dip at the inflexion point of about $1^\circ$). The fold shapes are numerically simulated single-layer folds for a viscosity contrast of 50. The black line represents the fold amplification of all viscous single layers, because any particular value of shortening (bottom axes) can be calculated from the value of the scaled stretch (top axes) through the crossover strain ($e_C$), which corresponds to a particular value of initial amplitude and growth rate. Values of the scaled stretch of 1 and 1.25 are used to separate three folding stages, namely, nucleation, amplification and kinematic growth (see Fig. 4).
The value of $A/\lambda$ increases with increasing values of $S$ and the curve $A/\lambda$ versus $S$ is referred to as an amplification curve (Fig. 2). For different values of the initial growth rate and the initial ratio of amplitude to wavelength, the growth of $A/\lambda$ versus $S$ is considerably different (Fig. 2A). However, if $A/\lambda$ is plotted versus $S_S$ instead of $S$ then all different amplification curves fall onto essentially a single curve (Fig. 2B). This means that the amplification of most single-layer folds for different values of initial growth rates (or alternatively different values of viscosity contrast) and initial amplitudes can be accurately described by a single amplification curve, especially for values of $A/\lambda$ larger than around 0.05 (which corresponds to a limb slope of about 17° at the inflexion point). For values of $A/\lambda$ larger than 0.05 the maximal variation in limb dip for all different curves is less than 5° at a fixed value of $S_S$.

The amplification curve for an initial growth rate of 15 (corresponding to a viscosity contrast of about 50; Fletcher, 1977) and an initial ratio of amplitude to wavelength of 3e−3 (corresponding to an initial limb slope of about 1°) is used as a representative amplification curve (Fig. 3). For the amplification curve two coordinate systems are displayed: (i) the system limb dip versus bulk shortening, and (ii) the system $A/\lambda$ versus $S_S$. In the system $A/\lambda$ versus $S_S$ the curve represents the universal amplification curve valid for all viscous single-layers having different initial amplitudes and viscosity contrasts. The displayed fold shapes are numerically calculated for a viscosity contrast of 50 and the layer initially exhibited a sinusoidal shape corresponding to the dominant wavelength.

The curve shown in Fig. 3 has been plotted using Eqs. (7) and (9). Eq. (9) is an implicit equation for the ratio of $A/\lambda$. In order to plot the curve $A/\lambda$ versus $S_S$, first an array of values of $A/\lambda$ is specified (e.g., $A/\lambda=[0.01, 0.02, \ldots, 0.4]$) and the corresponding values of $S_S$ are calculated for specific values of $\alpha_0$ and $A_0/\lambda_0$. Then, the two corresponding arrays containing values of $A/\lambda$ and of $S_S$ are used for plotting the curve $A/\lambda$ versus $S_S$. These two arrays can further be used to plot the ratios of $A/A_0$, $\alpha/\alpha_0$, and $V/V_0$ versus the ratio of $\lambda/\lambda_0$ (Fig. 4) where $\alpha$, $V$ and $V_0$ are the instantaneous fold growth rate, the instantaneous amplification velocity and the initial amplification velocity, respectively. The instantaneous growth rate is calculated using a discretized version of Eq. (1), $\alpha \approx \ln (A^{t+\Delta t}/A^t)/\Delta t - 1$, and the amplification velocity is calculated using $V = \partial A/\partial t \approx (A^{t+\Delta t} - A^t)/\Delta t$, where $t$ represents a certain index of the array storing the amplitudes, $\Delta t$ is the strain interval and $\Delta t$ is the time interval between two consecutive amplitude values. During amplification, the instantaneous growth rate is decreasing with progressive shortening and becomes zero for a value of $\lambda/\lambda_0$ about 1.25 (Fig. 4B). If the fold growth rate is zero, then the folding instability is not active anymore and the fold growth is due to kinematic amplification only, which is caused by the progressive pure shear shortening. The amplification velocity shows a maximum for a value of $\lambda/\lambda_0$ equal to about one (Fig. 4C). For different values of the initial growth rate and the initial ratio of amplitude to wavelength the different curves for $\alpha/\alpha_0$ versus $\lambda/\lambda_0$ and $V/V_0$ versus $\lambda/\lambda_0$ do not fall onto a single curve. However, for all curves the ratio of $\alpha/\alpha_0$ is always zero for a value of $\lambda/\lambda_0$ about 1.25 and the ratio of $V/V_0$ always exhibits a maximum for $\lambda/\lambda_0$ about one. Therefore, the values of $\lambda/\lambda_0$ equal to 1 and equal to 1.25 are chosen to define the boundaries between three folding stages of viscous single-layers: Nucleation for $\lambda/\lambda_0<1$, amplification for $1<\lambda/\lambda_0<1.25$ and kinematic growth for $1.25<\lambda/\lambda_0$. The boundary between nucleation and amplification defines the point where the amplification velocity exhibits a maximum and the boundary between amplification and kinematic growth defines the point when the growth rate due to the mechanical folding instability vanishes. However, due to the approximations in the analysis and the fact that the scaling does not provide exactly one single curve, the boundaries between the folding stages should not be considered as sharp boundaries but rather as transition zones (Fig. 4).

3. Numerical verification

The amplification Eq. (6) has been verified numerically for the amplification of single folds, which initially exhibited a sinusoidal shape corresponding to the theoretical dominant wavelength (Schmalholz and Podladchikov, 2000). In this section, Eq. (6) is tested numerically for the amplification of single folds that develop within evolving fold trains generated by shortening of single layers with initial random geometrical perturbations of the layer interfaces (Fig. 5).

The applied self-developed numerical algorithm solves the two-dimensional Stokes equations for incompressible viscous flow and is described in the Appendix. All model boundaries are kept straight during the pure shear deformation and free slip boundary conditions are applied on all boundaries. One horizontal and one vertical boundary are kept fixed while the other horizontal and vertical boundaries are extended and shortened by the pure shear velocity field, respectively. For the numerical simulations 501 nodes (250 elements) in the horizontal direction, 113 nodes (56 elements) in the vertical direction (25 nodes across the layer) and 500
time steps are used. Between two consecutive time steps the incremental horizontal bulk shortening was between 0.1% and 0.2%.

Numerical simulations have been performed for viscosity contrasts of 30, 50 and 60 (Fig. 5). The layers have initially a length to thickness ratio corresponding to nearly 15 times the theoretical dominant wavelength to thickness ratio. Two different random amplitude perturbations have been superposed on both the initially flat upper and lower layer interfaces. The random perturbation was generated by creating a random array with the same number of random values as nodal points in the horizontal direction. The values of the random array have been summed cumulatively and then the linear trend of the array has been subtracted, which yields a perturbation where the long wavelength components are more pronounced (see Mancktelow, 2001, for the impact of initial geometrical noise on single-layer folding). The random perturbation was generated using the MATLAB (© 1994–2005 The MathWorks, Inc.) functions rand, cumsum and detrend. The maximal amplitude of the random perturbation corresponds to about 0.02 of the initial layer thickness.

In each numerically calculated fold train four folds have been selected and separated through their convex downward fold hinges (Fig. 5). The individual folds are neither regular nor periodic, and the distance between the fold hinges does not correspond to a wavelength in the mathematical sense (Fletcher and Sherwin, 1978). However, for the purpose of quantifying the amplification of the individual folds, the horizontal distance between two convex downward fold hinges is used as a measure of the wavelength. The amplitude has been determined by the average vertical distance of the middle line of the convex upward and the two convex downward hinges (Fig. 1). The locations of the final fold hinges have been fixed to the layer and the fold evolution has been restored using the numerical results of the previous time steps. In Fig. 6 the fold amplification is plotted for the four folds corresponding to a viscosity contrast of 50 (Fig. 6A), 30 (Fig. 6B) and 60 (Fig. 6C). In addition, the amplification Eq. (6) is plotted for the growth rate value...
corresponding to the analytically predicted maximal growth rate (Fletcher, 1974) for a viscosity contrast of 30, 50 or 60. The initial ratio of amplitude to wavelength used in the amplification equation is the average of the initial ratios of amplitude to wavelength of the four folds. All individual amplification curves are close to the

Fig. 5. Numerically simulated single-layer fold trains for viscosity contrasts of 50 (A), 30 (B) and 60 (C). Four folds have been selected from each fold train. The individual folds have been separated by the convex downward fold hinges. Above each individual fold the ratio of fold arclength to layer thickness ($L/H$) is displayed.
analytically predicted amplification curve. Scaling the amplification curves by the theoretical value of the crossover strain collapses all amplification curves for the 12 folds having three different viscosity contrasts onto a narrow band of amplification curves (Fig. 6D). For each of the 12 numerically calculated folds the boundaries between the three folding stages can be determined using the values of the scaled stretch of 1 and 1.25 (Fig. 6D).

The analytically predicted growth rate versus wavelength curve (Fletcher, 1974, 1977) has been tested by the discrete Fourier transform of the middle line of the fold train for a viscosity contrast of 50. The discrete Fourier transform has been calculated using the MATLAB `fft` function (Fig. 7). The power spectrum of the discrete Fourier transform is plotted versus the corresponding ratios of Fourier wavelength to layer thickness at three different stages (Fig. 7A to C). For small shortening, the power spectrum represents the initial random perturbation showing increasing powers with increasing wavelengths (Fig. 7A).

The growth rate versus wavelength curve of the entire layer at a given amount of shortening can be generated directly from two power spectra corresponding to two consecutive numerical time steps using the approximation (valid for small time steps)

$$
\alpha \approx \ln \left( \frac{P_t + \Delta \lambda}{P_t} \right) / \Delta t - 1
$$
where \( P, t \) and \( \Delta t \) are the power spectra, the simulation time corresponding to a certain amount of shortening and the time interval between two numerical time steps, respectively. The resulting growth rate spectrum is smoothed to filter out errors caused by the orders of magnitude differences in the power spectrum. The smoothing algorithm employed is

\[
\alpha(i) = \alpha(i) + 0.4[\alpha(i+1) - 2\alpha(i) + \alpha(i-1)]
\]

where \( i \) is the index of a certain wavelength. The smoothing algorithm is similar to the finite difference form of a diffusion equation and guarantees that the area under the growth rate curve is conserved during the smoothing.

At a shortening of 17%, the growth rate spectrum derived by the Fourier power spectrum agrees well with the analytical growth rate spectrum and at 34% shortening, the analytical growth rate spectrum overestimates the actual growth rates, which indicates the failure of the linear stability analysis predicting exponential amplification with a constant growth rate. The main characteristics of the Fourier power spectra and growth rate spectra for the fold train with a viscosity contrast of 50 are similar to the Fourier power spectra and growth rate spectra for the fold train with a viscosity contrast of 30 and 60.

### 4. Application

The average value of \( A/\lambda \) which corresponds to a value of \( S_S = 1 \) is around 0.05 (i.e. a limb dip at the inflexion point of about 17\(^\circ\) for a fold with sinusoidal shape) and the average value of \( A/\lambda \) corresponding to a value of \( S_S = 1.25 \) is around 0.15 (i.e. a limb dip of about 43\(^\circ\) for a fold with sinusoidal shape, Fig. 2B). Therefore, folds exhibiting values of \( A/\lambda < 0.05 \) reside in the nucleation stage, folds exhibiting values of 0.05 < \( A/\lambda \)
λ<0.15 reside in the amplification stage and folds exhibiting values of 0.15<A/λ reside in the kinematic growth stage (Fig. 3). The measured value of A/λ of a fold allows immediately concluding if, for example, the amplification velocity is still increasing or is already decreasing, or if the fold will maintain a constant arclength during future amplification, as it would be the case if it is in the kinematic growth stage. Note, the values of 0.05 and 0.15 for A/λ, which define the boundaries between the three folding stages, are not exact boundaries but, instead, estimates for the location of the transition from one folding stage to the next.

The scaled amplification equation is furthermore applied to a folded single-layer presented in Lisle (2003) (Fig. 8). The geometry of one particular fold has been drawn from the picture and the three parameters approximating the amplitude, wavelength and thickness have been measured (inset of Fig. 8). These three parameters allow estimating the horizontal shortening accumulated during folding using the strain estimation method proposed by Schmalholz and Podladchikov (2001), which yields a bulk shortening of about 62%. The amplitude to wavelength ratio of this fold is about 0.45 and has been plotted onto the scaled amplification curve. The fold plots in the kinematic growth stage (Fig. 8). From the ratio of A/λ alone, the amount of shortening cannot be determined. However, the amount of shortening accumulated during kinematic growth can be calculated, i.e. the amount of shortening accumulated between 0.15<A/λ<0.45. All folds amplify identically during kinematic growth (Fig. 2B). The arclength remains nearly constant during amplification for folds having high viscosity contrast (≫ 100; Schmalholz and Podladchikov, 2001). Folds having small viscosity contrasts exhibit the same amplification curve than folds having high viscosity contrasts and, therefore, all folds exhibit the same growth of A/λ versus S during kinematic growth. The values of A/λ control the arclength through Eq. (7), and, hence, all folds amplify with an identical arclength evolution, which means with a nearly constant arclength during kinematic growth. In the example the ratio of L/λ can be calculated from the

Fig. 8. Application of the scaled amplification equation to a single-layer fold train presented in Lisle (2003). A single fold has been selected and drawn from the picture, and is shown in the inset of the photo from Lisle (2003). The amount of horizontal shortening accumulated during kinematic growth can be estimated using the measured value of A/λ (see text).
measured $A/\lambda$ using Eq. (7), which yields $L/\lambda$ about 2.24. The value of 1.25 of the scaled stretch corresponds to a value of $A/\lambda$ about 0.15 and hence a value of $L/\lambda$ about 1.21. Since $L$ is constant we can use the equations

$$\frac{L}{\lambda_1} \bigg|_{x=0.45} = 2.24, \quad \frac{L}{\lambda_2} \bigg|_{x=0.15} = 1.21, \quad \frac{L}{\lambda_2} = \frac{\lambda_2}{\lambda_1},$$

$$= \frac{2.24}{1.21} \quad e = 1 - \frac{\lambda_1}{\lambda_2} = 0.46$$

During folding between the values of $A/\lambda=0.15$ and $A/\lambda=0.45$ the horizontal hinge distance shortened about 46% (Fig. 8).

5. Discussion

The scaled amplification equation is an approximation describing the growth of the dimensionless fold amplitude with increasing values of the scaled stretch for viscous single-layer folding. The main benefits of this equation are its general applicability to all viscous single-layer folds with varying initial amplitude and viscosity contrasts, and its simplicity. Already existing analytical solutions, even the extensive third order analysis presented in Johnson and Fletcher (1994), are only valid for limb slopes up to about 30° and arc, furthermore, considerably more complex. The calculation of high fold amplitudes requires usually a numerical integration in time while the presented equation provides an analytical relation valid up to amplitudes corresponding to limb slopes of about 70°. The applicability of the scaled amplification equation has been verified by numerical solutions of the full set of the governing continuum mechanics equations describing incompressible viscous flow in two dimensions. The scaled amplification equation can be regarded as a first order approximation to single-layer fold amplification, which captures the fundamental mechanical processes active during folding. More exact solutions show, for example, that the upper and lower fold interfaces exhibit different shapes or that the dominant wavelength should change during the wavelength selection process caused by horizontal shortening (Sherwin and Chapple, 1968; Johnson and Fletcher, 1994). Therefore, to obtain more accurate information on the exact fold shape or the exact stress and strain distribution within the single-layer fold, more elaborated analytical or full numerical solutions are required. However, for describing the first order evolution of the fold amplitude versus horizontal shortening, the presented amplification equation is sufficient.

The vertical and horizontal hinge distances of natural folds are the easily measured parameters specifying the fold shape and are less sensitive to measurement errors than the hinge curvature. Therefore, these hinge distances are of particular interest for structural geologists interested in the amplification history of folds. This study shows that the horizontal and vertical hinge distances are good approximations for the mathematical parameters wavelength and amplitude, respectively, in the context of describing fold amplification. This is valid, although single folds are neither regular nor symmetric (Fig. 5).

The ratio of arclength to layer thickness is sometimes used to estimate the initial wavelength to thickness ratio of folds in order to infer the viscosity contrast between layer and matrix (Sherwin and Chapple, 1968). The numerical simulations in this study show that the ratios of arclength to layer thickness of single folds within fold trains can vary considerably, even for the considerably simple model setup employed in this study. Besides, different rheologies such as power-law or viscoelasticity and the effects of an anisotropic matrix can provide the same dominant wavelength for considerably different effective viscosity contrasts (Biot, 1965; Fletcher, 1974; Schmalholz and Podladchikov, 1999). Therefore, estimates of effective viscosity contrasts based exclusively on the ratio of fold arclength to fold thickness are only suitable to a limited extent.

6. Conclusions

The scaled amplification Eq. (9) yields essentially a single curve describing the amplification of viscous single-layer folds for different values of viscosity contrast and initial amplitude (Fig. 2). The scaled amplification equation is applicable for single folds with realistic fold shapes that developed within fold trains (Fig. 5). Although the individual folds are irregular, not periodic and vary in fold shape, their amplification curve is close to the amplification curve predicted by the amplification equation (Fig. 6). The vertical and horizontal hinge distance of natural folds are good approximations for the mathematical parameters amplitude and wavelength, respectively, which are used in analytical solutions for folding. The transition zones between, successively, nucleation, amplification and kinematic growth can be quantified by two values of the scaled stretch and are generally valid for viscous single-layer folding (Figs. 3 and 4). During folding of any viscous single-layer, the maximal amplification velocity is reached at values of the scaled stretch about 1 and the growth rate due to the mechanical buckling instability vanishes at a value of the scaled stretch about 1.25. The scaled amplification Eq. (9) together with the growth rate equation (Biot, 1961; Fletcher, 1974) represents the suitable mathematical framework to describe the main mechanical characteristics of finite amplitude buckling of single viscous layers.
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Appendix A

The finite element algorithm used in this study is summarized below. The conservation equations for slow incompressible flow are slow incompressible flow in the absence of body forces in two dimensions are (e.g., Bathe, 1996; Haupt, 2002):

Conservation of mass

\[ \frac{1}{K} \frac{\partial p}{\partial t} = - (\nabla^T \cdot \vec{v}) \]  

(A1)

Conservation of linear momentum

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \]

\[ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \]  

\[ \frac{\partial p}{\partial t} = 0 \]  

(A2)

where \( \sigma_{xx} \) and \( \sigma_{yy} \) are components of the total stress tensor in the x- and y-direction, respectively, \( \sigma_{xy} \) is the shear stress, \( p \) is the pressure, \( \vec{v} \) is a vector containing velocities in the x- and y-direction (\( v_x \) and \( v_y \), respectively), \( K \) is the compressibility parameter and \( \nabla^T \) is the divergence operator. Eq. (A2) deviates from the standard form for incompressible flow (i.e., \( \nabla^T \cdot \vec{v} = 0 \)), but is only applied for very large values of \( K \), so that the resulting divergence of the velocity field goes to zero, which means to \( 10^{-15} \) in this study. Application of Eq. (A2) is often referred to as the penalty approach for incompressible flow (e.g., Cuvier et al., 1986; Hughes, 1987). The constitutive equations for a linear viscous rheology are:

\[ \vec{\sigma} = -p \vec{m} + D \vec{\dot{v}} \]  

(A3)

where

\[ \vec{\sigma} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix}, \quad \vec{m} = \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}, \quad \vec{\dot{v}} = \begin{Bmatrix} \frac{\partial v_x}{\partial x} \\ \frac{\partial v_y}{\partial y} \\ \frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \end{Bmatrix}, \quad D = \begin{bmatrix} 4/3 \mu & -2/3 \mu & 0 \\ 2/3 \mu & 4/3 \mu & 0 \\ 0 & 0 & \mu \end{bmatrix} \]

(A4)

with \( \mu \) the Newtonian viscosity. Discretization of the governing equations and numerical integration is performed using the isoparametric Q9/3-element with 9 nodes for the biquadratic continuous velocity degrees of freedom and 3 for the linear discontinuous pressure degrees of freedom (Hughes, 1987). Discretization leads to the system of equations (e.g., Hughes, 1987):

\[ \begin{bmatrix} K \\ -K \Delta t G \end{bmatrix} \begin{bmatrix} \vec{\ddot{v}} \\ \vec{p}^{new} \end{bmatrix} = \begin{bmatrix} 0 \\ M \vec{p}^{old} \end{bmatrix} \]  

(A5)

where swung dashes denote vectors containing nodal values of the respective variables. The time derivative in Eq. (A2) has been replaced by a finite difference quotient with \( \Delta t \) being the time step (\( \partial p/\partial t \approx (p^{new} - p^{old})/\Delta t \)). The three matrices \( K, G \) and \( M \) are:

\[ K = \int \int B^T DB \, dx \, dy, \]

\[ G = - \int \int B_G^T N_p \, dx \, dy, \]

\[ M = \int \int N_p^T N_p \, dx \, dy \]  

(A6)

where vector \( N_p \) contains the pressure shape functions and matrix \( B \) and vector \( B_G \) contain spatial derivatives of the velocity shape functions in a suitable organized way (Zienkiewicz and Taylor, 1994). The integrations are performed numerically using nine integration points per element. Using discontinuous pressure shape functions allows eliminating the pressure on the element level. This elimination leads to a system involving only unknown velocities:

\[ \vec{L} \vec{\ddot{v}} = - G \vec{p}^{old} \]  

(A7)

where

\[ \vec{L} = K + K \Delta t G M^{-1} G^T \]  

(A8)

Values of \( \vec{p}^{new} \) are restored during the Uzawa-type iteration algorithm, during which Eq. (A7) is solved iteratively with updated values of \( \vec{p}^{old} \) until the divergence of the velocity converges towards zero (i.e. \( 10^{-15} \), e.g., Pelletier et al., 1989). After every time step, the resulting velocities are used to move the nodes of each element with the displacements resulting from the product of velocities times time step. Then, the new velocities are again calculated for the new grid.
The presented finite element algorithm has been successfully tested twice. First, by correctly reproducing the analytical growth rate (Fletcher, 1977) for viscous single layer folding, and second, by correctly reproducing the analytical pressure field around a rigid inclusion (Schmid and Podladchikov, 2003).

References