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Estimation of Electoral Disproportionality and Thresholds via MCMC[1]

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ABSTRACT

For statistical as well as political reasons combined measures of electoral disproportionality and electoral thresholds are necessary in order to adequately summarize electoral institutions. With few exceptions, none of these quantities can be reliably inferred directly from the provisions of the electoral law, thus impairing "large scale" comparative studies. Through the use of sampling based Bayes methods we are able to simultaneously estimate thresholds and disproportionality from electoral returns. We apply the proposed procedure on 45 electoral systems in use over 216 election to the national parliament in the 15 countries of the European Union in the period 1945-1996. The resultant two-dimensional summary of electoral systems has several advantages over measures of disproportionality currently used in comparative politics.

Keywords: Electoral Disproportionality, Electoral Thresholds, Gibbs Sampling, Markov Chain Monte Carlo, Metropolis Algorithm.

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[1] I thank Gary Cox, Ken Scheve, Alastair Smith, and Jim Vreeland for helpful comments.

1. INTRODUCTION

The measurement of electoral disproportionality figures prominently in the agenda of electoral systems research. The interest in disproportionality emanates both from a desire to quantify the performance of electoral systems against a normative benchmark of fair or proportional allocation, but also because of the influence of this phenomenon in shaping the party system.

One of the established tenants of this research is that there is no unique, universally accepted way to measure disproportionality (Gallagher, 1991). This is a direct consequence of the fact that measures of disproportionality attempt to condense into a single dimension what is essentially a multi-dimensional phenomenon[2]. It is thus unavoidable that measures of disproportionality involve some loss of information and alternative measures reflect or ascribe different valuations as to the aspects of the seat allocation process that should weigh more importantly in quantifying the phenomenon.

As Cox and Shugart, 1991, argue and the practice of many authors reveals, there is room for a positive research to quantify disproportionality despite the subjective nature of the normative criteria that are embodied in different measures. The goal of this research program is to achieve measurement that at a minimum satisfies some more objective desiderata. In this paper we aim to contribute in exactly this direction.

Our contribution is based on two premises. First, we claim that both the underlying constancy of electoral institutions over several elections, as well as the nature of the allocation process in most existing national electoral systems requires and permits the *statistical estimation* and not the *measurement* of disproportionality. By using statistical estimation we can capture the constant, systematic component of the seat allocation induced by electoral systems. In that direction, and building on a number of previous contributions in political methodology, we develop a statistical model to represent the process of translating vote shares to seat shares. We obtain our measures of disproportionality by estimating the

[2] In the presence of n contesting parties, the seat and vote shares that constitute the data used to measure disproportionality lie in a $2(n-1)$ dimensional space.

parameters of this statistical model.

Our second innovation over traditional approaches of measuring disproportionality and the political methodology literature is that we opt for a specification of this model that accounts for two forms of (dis)proportionality of electoral systems: severe disproportionality for small parties due to system-level electoral *thresholds*, and what we will term *weak (dis)proportionality* for parties above thresholds. We choose to jointly estimate (weak) disproportionality and thresholds for both practical and theoretical reasons.

First, we argue that unidimensional measures of (dis)proportionality provide an unsatisfactory summary of the phenomenon. The essence of the problem lies in the bifurcated nature of disproportional allocations induced when thresholds are positive. In such cases the expected seat share of parties that fall below the threshold is zero by definition, irrespective of the degree of (dis)proportionality for parties above thresholds. We consider this severe form of disproportionality introduced by thresholds a significant political attribute of an electoral system with potentially distinct consequences from milder forms of disproportionality for parties above thresholds. Our estimation procedure preserves this conceptual distinction and allows for the direct, simultaneous quantification of each of these two forms of disproportionality.

This is clearly not a property shared with traditional indices, which are based on deviations of realized seat shares from "ideal" PR allocations[3]. Besides being a politically more relevant depiction of the strategic forces induced by the electoral system, our approach produces more robust[4] measures compared to these alternatives. The latter fail the robustness criterion because deviations – and measured disproportionality – vary significantly from

[3]See Gallagher, 1991, 1992, Cox and Shugart, 1991, Lijphart, 1994, ch. 3, Penades, 1997, and Penisi, 1998, for a detailed review as well as enlightening discussion of the theoretical justifications and properties of these measures.

[4]By robust we mean a measure that does not radically change values for slightly different electoral outcomes. This is different from the usage of the term in Pennisi, 1998, whose robustness criterion is whether allocation formulas consistently perform better than others according to alternative indices of disproportionality.

election to election depending on the number of parties that happen to exceed the threshold. In essence, "ideal PR" allocations in these cases are not simple but relative vote shares – relative to the set of parties that exceed the electoral threshold. For example the German electoral system is highly proportional for all parties that receive more than 5%, *i.e.* these parties receive a PR share based on their relative vote shares.

One could argue that this deficiency of disproportionality indices does not warrant the computationaly demanding alternative we propose. For example, if electoral thresholds are known[5], these indices can be redefined to reflect the appropriate notion of deviation from ideal PR allocations, *i.e.* using relative vote shares for parties above thresholds. We argue that this alternative is not feasible or appropriate for two reasons.

First, electoral thresholds are rarely known with certainty nor can they be inferred solely from the provisions of the electoral law[6]. Even when explicitly instituted, thresholds may apply at various levels (tiers) of seat allocation, or be compromised by additional provisions[7] which render system-wide thresholds an unknown, variable quantity across elections. Imputations, such as in Taagepera, 1989, 2001, Taagepera and Shugart, 1989, and Lijphart's 1994, ch. 2 "effective threshold" are credible only in certain families of electoral systems. These approximations require a number of simplifications and *ad hoc* assumptions that reduce the comparability of these estimates across systems. For example, calculating thresholds on the basis of average district magnitude results to overestimation since one large district – as is the case for metropolitan districts in Spain, Greece, etc. – is sufficient to allow representation for small parties. Ultimately, effective thresholds depend on the distribution of party support across districts and over time[8], a variable rarely explicit in the electoral

[5]There is also the possibility that electoral thresholds are zero, but this is a rather uninteresting theoretical case given the finite composition of representative bodies.

[6]Exceptions include the two post WWII Dutch systems or the current (since 1993) Greek system that provide for unconditional system-wide electoral thresholds.

[7]For example, the German system for elections in the Bundestag provides that the 5% national threshold be ineffective for parties that receive more than 3 MPs from single member constituencies.

[8]Provided their electoral support is concentrated in a particular district, even small parties can achieve representation.

law.

Second, even if nation-wide thresholds are known, disproportionality indices constitute less desirable alternatives compared to our weak (dis)proportionality estimates. We elaborate on this point in section 4, but our claim is based on the following three arguments. First, many of the deviations-based indices fail to capture the "political character of disproportionality," a point raised by Cox and Shugart, 1991. Second, our disproportionality parameter is a more accurate reflection of the *systematic deviation* of the electoral system from ideal PR allocation, while alternative indices *conflate this systematic component with random fluctuation around systematic or expected seat allocations.* Finally, our procedure also produces an honest, system-specific summary of the uncertainty that can be placed on our estimates, which can be used for purposes of statistical inference or can be appropriately incorporated in studies where these measures are used as explanatory variables.

Due to the introduction of electoral thresholds, the statistical model we specify is not amenable to conventional estimation techniques. Instead, we use sampling based Bayesian methods. We apply this estimation procedure on electoral data from 216 elections to the national parliaments of EU countries in the period 1945-1996 and we estimate national threshold and disproportionality parameters for 45 electoral systems.

We are now ready to motivate our statistical model of the electoral system and relate it to previous political methodology literature. We do so in section 2. In section 3, we detail the application of the derived estimator on data from national elections in EU countries. We discuss the results in section 4, focusing on the advantages of our estimates over more traditional approaches to the calculation of disproportionality in comparative politics. We conclude in section 5.

2. PROPORTIONALITY AND ELECTORAL THRESHOLDS

Consider a set of N parties whose vote share in each of M elections is given by:

$$v_{ij} \geq 0, i = 1, ..., N \text{ and } j = 1, ..., M. \tag{1}$$

In addition to being non-negative, the vote shares above also satisfy

$$\sum_{i=1}^{N} v_{ij} \leq 1, j = 1, ..., M. \tag{2}$$

Notice that, though allowed, equality is not required in (2) for reasons that will be made explicit in short. This is particularly convenient for our purposes because electoral data typically involve some loss of information due to the fringe parties whose vote share is reported under the "others" category. To each of the vote shares in (1) the electoral system assigns a number of seats denoted by

$$s_{ij} \geq 0, \sum_{i=1}^{N} s_{ij} = S_j, i = 1, ..., N \text{ and } j = 1, ..., M. \tag{3}$$

As outlined in the introduction, the approach in this analysis is to represent the electoral system as a non-degenerate probability distribution over seats. This is also necessary in the bulk of electoral systems at the national level since, except under very special circumstances[9], more than one seat allocations are possible for the same vote shares. A particularly attractive restriction on the possible electoral rules considered can be imposed by requiring that the expected seat share of party i, $E\left[\frac{s_{ij}}{S_j}\right]$, satisfies:

$$E\left[\frac{s_{ij}}{S_j}\right] = q_{ij} \equiv \frac{v_{ij}^{\alpha}}{\sum_{i=1}^{N} v_{ij}^{\alpha}}, \alpha \in [0, +\infty), i = 1, ..., N \tag{4}$$

Among the (infinite) probability distributions consistent with (4) we assume, following King, 1990, that

$$(s_{1j}, ..., s_{Nj}) \sim Multinomial\left[S_j; q_{1j}, ..., q_{Nj}\right], j = 1, ..., M. \tag{5}$$

The family of electoral rules in (4) has been postulated at least since Henri Theil, 1969[10]. The "cube law" for two-party systems with single member districts plurality electoral system is a special case ($N = 2$ and $\alpha = 3$) proposed at the beginning of the century

[9]Exceptions include electoral systems with a single district and a deterministic allocation formula that depends only on vote shares.

[10]To be precise, Theil suggests that the seat share of parties be exactly equal to this quantity, not just in expectation as is assumed here.

(Kendal and Stuart, 1950)[11]. When $\alpha = 1$, (4) implies simple Proportional Representation (PR), while values of α greater than 1 induce disproportional allocations that favor larger parties, and the converse is true when $\alpha < 1$. Hence, *α serves naturally as an index of (dis)proportionality*[12]. Notice that relations (4) and (5) hold for any subset of N parties as long as they hold for all the parties in the system, *i.e.* for N such that the inequality in (2) binds. Thus, as already alluded above, estimators of α derived on the basis of these assumptions retain their remaining properties even if data for a subset of parties $N > 1$ are used – although they are obviously inefficient.

Additional reasons suggest that deviations from proportionality satisfy (4). In particular, Theil shows that for non-trivial voting outcomes the allocation in (4) uniquely minimizes the deviation of vote shares from seat shares for fixed levels of overall deviation, where measures of deviation are drawn from information theory (Theil, 1969, Theorem 1). Put in other words, when the allocation of seats deviates from PR at some fixed level, the least "surprising" seat allocation is the one that satisfies (4). Although we will defend the qualified use of (4) throughout this study, it is important to emphasize that Theil's argument is not a statement about the real world: actual electoral systems may, and typically do deviate from this normative benchmark.

One such deviation identified early in the literature involves differential treatment of parties by the electoral system in the form of *partisan bias* (Tufte, 1973, Grofman, 1983, King and Browning, 1987, King, 1990, and the references therein). Partisan bias is an important political reality that arises from the peculiarities of the concentration of electoral power of political parties as well as districting practices. Although our model can be easily extended to incorporate electoral bias issues, our concern is with a more obvious violation of the seats-votes relation in (4) due to electoral thresholds[13].

[11]See Schrodt, 1981, and Taagepera, 1986, for additional references on the "cube law."

[12]Although in the political methodology literature α is also referred to as responsiveness (King, 1990), we prefer Theil's, 1969, p. 524 term of weak (dis)proportionality to emphasize the antithesis with the severe disproportionality induced by electoral thresholds.

[13]Because of its party-specific interpretation, partisan bias is less relevant for the purposes of comparative

In particular, let the threshold effective in election j be denoted by τ_j. Unless there is information to the contrary, we assume thresholds differ across elections due to the chance configuration of party strengths in different districts as well as other provisions of the electoral law that qualify their applicability. The analysis extends trivially to the case the threshold is common in all elections, *i.e.* $\tau_j = \tau, j = 1, ..., M$. Unlike Theil's weak (dis)proportionality parameter α, τ_j induces a severe form of disproportionality against small parties, since:

$$E[s_{ij}] = 0, \text{ if } v_{ij} < \tau_j, i = 1, ..., N \text{ and } j = 1, ..., M. \tag{6}$$

In account of (6), a modification of the seats-votes relation in (4) suggests itself:

$$E\left[\frac{s_{ij}}{S_j}\right] = q_{ij} \equiv \frac{f(v_{ij};\tau_j)^\alpha}{\sum_{i=1}^N f(v_{ij};\tau_j)^\alpha}, \alpha \in [0, +\infty) \tag{7}$$

where $f(\bullet;\bullet)$ is defined as:

$$f(x;\tau_j) = \begin{cases} 0, & x \in [0, \tau_j) \\ x, & otherwise \end{cases} \tag{8}$$

In combination, equations (5), (6), and (7) capture the two-geared disproportionality induced by most electoral systems: *severe disproportionality for small parties below the threshold, and (some) level of weak (dis)proportionality α for the remaining parties.*

Obviously, procedures that estimate the weak (dis)proportionality parameter α ignoring thresholds when the latter are positive result in overestimation. This is true of estimates of α based on equations (4) and (5), but applies equally to all indices of disproportionality widely used in comparative politics. Furthermore, the degree of this overestimation depends on the number of parties below the electoral threshold, thus significantly compromising the comparability of these estimates. Strategic behavior on the part of political actors further

work that assesses the effects of electoral institutions on the party system. Also, we believe the incorporation of thresholds to be a more relevant enrichment of the relation in (4) that should precede the introduction of partisan bias parameters. Under the false restriction that thresholds are zero bias estimates for parties below thresholds will be overestimated. Indeed, we can think of thresholds as a special form of bias that affects all parties below that vote share.

complicates matters since, as has been pointed out (Taagepera and Shugart, 1989:123, Lijphart, 1994:97, Cox, 1997:173-8), the number of parties likely to fall below the threshold may correlate with the weak (dis)proportionality of the system, resulting in systematically biased estimates across systems that take the form of a conservation of disproportionality[14].

As discussed in the introduction, if the thresholds τ_j are known then estimation of disproportionality is much more straightforward. For our purposes, if τ_j are not known, estimation of α can be arrived at by appropriate use of data for parties with positive seat share. For these, say, l parties a likelihood function can be derived consistent with equations (4) through (5) by conditioning on the event that $s_{ij} > 0$. But this is inefficient, since data may be discarded for those parties that happen to receive zero seats even though their vote share is above the corresponding threshold – this is more likely the higher α is. Most importantly, this alternative leaves unresolved the problem of the estimation of thresholds. As we are about to show, recent advances in sampling based Bayesian methodologies allow a straightforward solution for the simultaneous estimation of both the weak proportionality parameter α and the thresholds τ_j.

As a first step, we complete the specification of the model by assigning prior distributions on these parameters. We assume that thresholds, τ_j, are distributed independently[15] according to some common distribution truncated by zero to the left and 1 to the right:

$$\tau_j \sim [\tau_j] \,, \tau_j \in [0,1) \,, j = 1, ..., M \tag{9}$$

Though, as we discuss later, other choices are feasible and perhaps superior we assume a uniform prior for the thresholds. We work with a transform, δ, of the the weak proportionality parameter, α, such that:

$$\alpha = \frac{\delta}{1-\delta}, \delta \in [0,1) \tag{10}$$

[14]Although, the conservation of disproportionality does not solely depend on electoral thresholds. One aspect of this argument is that strategic voting in plurality systems will understate the vote share of loosing parties, thus reducing measured disproportionality even if these parties achieve nation-wide representation.

[15]The independence assumption cannot be relaxed without significantly increasing the complexity of associated calculations. We discuss this point at the end of this section.

to which we also assign a uniform prior

$$\delta \sim [\delta] = U[0,1). \tag{11}$$

Note that besides our prior assumptions, data suggest obvious logical restrictions on the possible values of thresholds. In particular, the realization of the party vote and seat shares in election j and equations (7), (8) allow us to deduce that the corresponding threshold τ_j is strictly smaller than the minimum vote share among parties that receives legislative seats, say $\overline{\tau}_j$, where:

$$\overline{\tau}_j = \min_{i=1,\dots,N} \{v_{ij} \mid s_{ij} > 0\} \tag{12}$$

Thus, equations (5), (7), and (11) determine a joint distribution of the data $\{\mathbf{s}_j = (s_{1j}, ..., s_{Nj})\}_{j=1}^{M}$ and the parameters δ, and $\tau_j, j = 1, ..., M$ which (incorporating the fact that $\tau_j < \overline{\tau}_j$) can be recognized as:

$$\left[\{\mathbf{s}_j, \tau_j\}_{j=1}^{M}, \delta\right] = \prod_{j=1}^{M} [\mathbf{s}_j \mid \tau_j, \delta]\,[\tau_j]\,[\delta] \propto \tag{13}$$

$$\propto \prod_{j=1}^{M}\prod_{i=1}^{N}\left(\frac{f(v_{ij};\tau_j)^{\alpha}}{\sum_{i=1}^{N} f(v_{ij};\tau_j)^{\alpha}}\right)^{s_{ij}}, \tau_j \in [0, \overline{\tau}_j), j = 1, ..., M, \delta \in [0,1)$$

Due to the awkward form of the likelihood specified by (5) and (7), it is impossible to directly obtain the posterior distribution of the parameters. But the posterior moments of δ (i.e. α) and τ_j can be calculated – up to arbitrary tolerance level – via Markov Chain Monte Carlo methods. In particular, to achieve sampling from the joint posterior we make use of the Gibbs sampler[16], which requires iterative sampling from Markovian updates of the full conditionals of the unknown parameters.

To this end, notice that from the joint distribution in equation (13) we can deduce:

$$\left[\delta \mid \{\mathbf{s}_j, \tau_j\}_{j=1}^{M}\right] \propto \prod_{j=1}^{M}\prod_{i=1}^{N}\left(\frac{f(v_{ij};\tau_j)^{\alpha}}{\sum_{i=1}^{N} f(v_{ij};\tau_j)^{\alpha}}\right)^{s_{ij}}, \delta \in [0,1) \tag{14}$$

[16]Geman and Geman, 1984. See Gelfand and Smith, 1990, Carlin and Louis, 1996, Tanner, 1996 for comprehensive presentations of these methods. For applications of the Gibbs Sampler in Political Science see Jackman, 1999, King, Rosen, and Tanner, 1999, Smith, 1997.

for δ and

$$\left[\tau_j \mid \delta, \{\mathbf{s}_j,\}_{j=1}^M\right] \propto \prod_{i=1}^N \left(\frac{f(v_{ij};\tau_j)^\alpha}{\sum_{i=1}^N f(v_{ij};\tau_j)^\alpha}\right)^{s_{ij}}, \tau_j \in [0, \overline{\tau}_j) \tag{15.j}$$

for each of the M thresholds. K cycles of iterative sampling from (14) and the M distributions in (15.j) result to a sample from the joint posterior of δ and τ_j as K tends to infinity. Upon obtaining a sample from the joint distribution via this process, estimates of the desired parameters can be obtained from an appropriate choice of sample size, say L, via simple Monte Carlo integration. For example, if τ_{jh} is the h-th realization of this sample, we can use the posterior mean[17], $\widehat{\tau}_j = \frac{1}{L}\sum_{h=1}^L \tau_{jh}$, as an estimate of the threshold, τ_j, while to estimate the overall (expected) threshold of the electoral system, τ, we can can calculate $\widehat{\tau} = \frac{1}{M}\sum_{j=1}^M \widehat{\tau}_j$. Some comments are in order:

Remark 1 *Under the uniform prior in (11) the conditionals in (15.j) are step functions where the number of steps is equal to the number of parties with vote share below* $\overline{\tau}_j$ *plus one.*

Remark **??** implies that thresholds are locally unidentified so that traditional methods such as maximum likelihood are inapplicable[18] in our problem. Related to Remark **??** is the following:

Remark 2 *If there do not exist parties with vote share below* $\overline{\tau}_j$*, the posterior of the threshold is given by* $\tau_j \sim U[0, \overline{\tau}_j)$.

Remarks 1 and 2 imply that the data contain additional information about thresholds beyond what is already reflected in the prior in (11) only if there exist parties with vote share below the logical upper bound in equation (12). This very crude usage of data can be significantly improved by adding a hierarchical structure on the prior distribution of the thresholds. This approach relaxes the assumption of independence by choosing a suitable parametric family for the distributions in (9) and priors for the parameters of this distribution.

[17]The median or the mode are equally feasible choices.

[18]Since the likelihood function has flat areas, the maximand is not unique.

Such a hierarchical structure on the threshold priors allows for the combination of information across elections, but involves non-trivial numerical issues due to the truncation imposed by the fact that $\tau_j \in [0, \overline{\tau}_j)$ (see Gelfand, Smith, and Lee, 1992, p. 525). In particular, sampling from the conditionals involved requires evaluation of the normalizing constant of the distribution in (9). Apart from computationaly expensive, such a procedure raises additional concerns relating to adequate numerical precision, hence we defer such extensions for future work.

3. ESTIMATION

In this section, we estimate the weak (dis)proportionality and threshold parameters for the electoral systems used in the period 1945-1996 for the election of national parliaments in the fifteen countries members of the European Union.

i. Data

Data on vote and seat allocations are from Mackie and Rose (1991, 1997) – the vote shares being calculated from the number of votes received by each party to improve accuracy. In using these data, it is important to keep in mind the task in hand, *i.e.* the estimation of parameters of electoral institutions. This requires adjustments in cases when representatives choose to identify themselves as members of separate parliamentary groups after their election under a common list. In such cases we corrected the data in order to maintain the original correspondence between the vote shares and allocated seats for the contesting parties or coalitions. As an example, the representatives of the Schleswig party elected in '73, '75, and '77 in Denmark were added to the representatives of the Centre Democrats with whom they were elected. This is also the practice reflected in the corrections reported in Lijphart, 1994, Appendix C.

Also, as already implied in the previous section, the "others" category was omitted from the data. This is not particularly efficient, but a discussion of procedures that could exploit the information from this category would lead us well beyond the scope of this study. One exception to the above rule involves the calculation of the upper bound of the electoral threshold, $\overline{\tau}_j$, in equation (12). If parties in the "others" category receive legislative representation, then the overall vote share of this group is included in the calculation of this upper bound as if it was a separate party.

Electoral systems were also identified on the basis of the information supplied by Mackie and Rose (and Lijphart, 1994, ch. 2). The resulting breakdown of systems correlates highly with Lijphart, 1994 (Lijphart's distinction is slightly more coarse). We also follow Mackie and Rose in identifying changes after 1990 (the endpoint of Lijphart's study), as well as the electoral systems in Greece prior to the 1967 military coup. It is straightforward to modify the model developed so far in order to incorporate a "change-point" analysis that would allow for a more rigorous statistical distinction of significant changes in electoral systems. For details on the usage of the Gibbs sampler in the context of change-point problems the reader may consult Carlin, Gelfand, and Smith (1992). The list of electoral systems and the election dates in which they were applicable can be found in Table 1. Overall, 45 electoral systems are estimated used in a total of 216 elections.

<<Insert Table 1 about here>>

ii. Implementation

In order to sample from the conditionals in (14) and (15.j), we make use of the Metropolis algorithm (Metropolis et al., 1953). This is reasonable in the case of (14) which is not directly available for sampling. The conditionals in (15.j) could be sampled from directly in light of Remark **??**, but this need not be more efficient than the chosen procedure. To implement the Metropolis algorithm, candidate variates for the thresholds were drawn from the uniform in $[0, \overline{\tau}_j)$. For the conditional in (14), we used the normal with mean the current variate, a standard deviation of 0.1, and truncation bounds at 0 and 1[19].

The Gibbs chain in which these Metropolis sub-chains are embedded converges for any

[19] To sample from this truncated normal we used the procedure described in Gelfand, Hills, Racine-Poon, and Smith, 1990, which is based on an exercise in Devroye, 1986 (exercise 10, p. 38).

choice of length for the Metropolis algorithm (Carlin and Louis, 1996:182, Tanner, 1996:181). Given the crude form of the conditionals in (15.j) a lengthy sub-chain seems awfully expensive and, as a consequence, we chose a single Metropolis step. We implemented the algorithm in the X-LISP environment[20]. Numerical precision is improved significantly by working with the logs of the conditionals – and the appropriate modification in the calculation of the Metropolis acceptance probability. For each of the electoral systems estimated, we run three parallel chains for $K = 4000$ iterations. In all cases, convergence occurs no later than $K = 200$, as is indicated in Figure 1 which superimposes these chains for selected parameters. The Gelman and Rubin (1992) convergence diagnostic corroborates this graphical evidence.

<<Insert Figure 1 about here>>

iii. Results

Results are summarized in Table 2. Along with the model developed in Section 2, we estimate a restricted model that is based on the assumption that the election specific thresholds are identically equal to zero. These are reported in the last two columns of Table 2. We choose to report posterior means. This is more or less justified in the case of the (transformed) weak proportionality parameter δ, the posterior of which is fairly Gaussian in shape[21]. The median is also a natural alternative, particularly for the thresholds, but no significant differences exist between the two. Finally, we have included in Table 2 Lijphart's (1994) estimates of the Effective Threshold in order to obtain an idea for the differences between the two procedures.

<<Insert Table 2 about here>>

[20] Software and data are available upon request.

[21] Note that under the restricted model and the uniform prior, the posterior mode is identical to the ML estimator.

4. DISCUSSION

Some conclusions as to the workings of the estimation procedure for electoral thresholds are immediate from Figure 1. In particular, the threshold displayed in Figure 1(b) is virtually uniform in the logical support of this parameter $[0, \overline{\tau}_j)$, *i.e.* the posterior is identical to the prior. This is not an instance of the possibility we describe in Remark **??**, since there were several parties with vote share below the logical upper bound defined in (12) in this election (UK2, 1950). Figure 1(b) simply suggests that it is virtually impossible to distinguish whether parties with vote share below $\overline{\tau}_j$ failed to achieve representation because of the high weak (dis)proportionality parameter δ (α), or because they failed to cross the implicit electoral threshold $\tau_j > 0$. Both alternatives are consistent with the highly disproportional patterns of seat allocation for parties above $\overline{\tau}_j$. Contrast this with the posterior in Figure 1(d) (GER3, 1969). Given the highly proportional allocation that the German system ensures for parties above the possible thresholds in $[0, \overline{\tau}_j)$, the data provide strong evidence that the threshold τ_j is higher than the vote share of the next smaller party to the one with vote share $\overline{\tau}_j$.

It is also apparent from Table 2 (Lijphart, 1994 is explicit about this) that Effective threshold approximations are completely unrealistic for majority/plurality systems (e.g. UK1, UK2, FRA3, FRA4, etc.) for which we estimate national thresholds considerably smaller than 35%. Clearly, the large number of districts in these systems and the concentration of local support in some of these areas permits representation for parties with much smaller vote share, as small as 0.63% for the second British system, UK2.

Effective threshold approximations also involve significant overestimation in PR systems where thresholds are inferred from district magnitude (e.g. FRA2 with 12.7% vs. our estimate of 2.60%,or IRE1 with 17% vs. our estimate of 1.11%) or other provisions (e.g. GRE6, GRE7, and GRE8 where the thresholds reported are those that apply at higher tiers of allocation). Effective thresholds are closest to the estimates obtained from the model in section 2 in the cases when the electoral system provides for explicit – but overall indeterminate – national thresholds. Such cases include all German systems (which also allows representation if more than three SMD representatives are elected) or the last Swedish system (SWE4, which allows parties national seat allocation if they exceed 12% in a single district despite falling below the 4% national threshold), etc.[22].

<<Insert Figure 2 about here>>

Of particular interest in evaluating the performance of the threshold estimates reported in Table 2 are the electoral systems in Table 1 for which known – uncompromised – thresholds are in effect at the national level. There are four such systems: DEN3, with 2%, NET1, 1%, NET2, 0.66...%, and GRE10, with 3%. In addition to the calculations reported in Table 2, we estimated the weak proportionality parameter under the correct restriction that election specific thresholds equal the known threshold of these four systems. The resultant posterior distributions along with the proportionality and threshold estimates under the unrestricted, agnostic models are displayed in Figure 2. Note that the dashed kernel density estimates from the correct model are only marginally distinguishable from those of the unrestricted model. Furthermore, the true threshold is always included within the center of mass of the posteriors from the unrestricted model[23]. In sum, the procedure performs particularly well in recovering both the true nation-wide threshold and the level of disproportionality above that level, despite the relatively vague, uninformative prior used in this study.

Additional evidence in favor of the proposed estimator is given in Figure 3, which displays the difference between estimates of weak proportionality from the restricted and unrestricted models against the threshold estimates from the unrestricted model. The resulting graph illustrates how procedures that estimate proportionality parameters assuming thresholds to be zero result to *overestimation of disproportionality*. As expected, the degree of overestimation is roughly, but not always, in proportion to the magnitude of the threshold. Deviation from the expected positive correlation of this relation in Figure 3 reflect the

[22]Lijphart calculates effective thresholds in these cases on the basis of the national threshold.

[23]Which are calculated from the combined threshold samples from all M election years.

sensitivity of proportionality measures under the restricted model to the number of parties that happen to be below the threshold.

<<Insert Figure 3 about here>>

As already discussed in the introduction, indices of disproportionality currently employed in comparative politics are equally sensitive to the number of parties below the electoral threshold. Thus, these measures are unreliable even in conveying a *ranking* of the disproportionality of different electoral systems, exactly because this ranking depends on how many parties fall below thresholds[24]. Also, as already discussed in section 2, weak (dis)proportionality measures are less sensitive to the size of the "other parties" category. Our estimates of the weak (dis)proportionality parameter from the unrestricted model have additional desirable properties over traditional indices, besides the superior performance in terms of robustness to the realization of the voting outcome or missing data.

<<Insert Table 3 about here>>

First, most disproportionality indices may fail to capture the "political character of disproportionality" (Cox and Shugart, 1991, p.350), *i.e.* the degree to which it favors or harms larger parties. In order to illustrate how traditional indices fail in that respect, consider the first two of the six hypothetical electoral systems displayed in Table 3a, and the corresponding values of the most widely used indices reported in Table 3b. Notice that, while both systems 1 and 2 receive identical levels of disproportionality according to these measures, the direction of deviation from proportionality is in favor of large parties in system 1 while it is in favor of small parties in system 2. This potential weakness of deviation-based indices was raised initially by Cox and Shugart, 1991, and was dismissed by Lijphart in his monograph (1994, p. 64-5). On the face of the examples in Table 3 his assertion that

[24]In table 3 we give an example illustrating the flip-side of this criticism. There two systems (5 and 6) that traditional measures deem as equally disproportional involve considerable difference in the induced pattern of seat allocation. System 5 has a thresholds between 25% and 35% but induces perfect PR allocation above that level, while system 6 is consistently a disrpoportional system.

deviation indices are free of such pitfalls can only be interpreted as an empirical statement, *i.e.* the fact that such examples, although possible, are not particularly relevant.

Even if this were true, the fact that traditional indices based on deviations are agnostic with regard to the direction of deviations from proportionality has an equally disturbing corollary: *traditional indices attribute "noise" in the allocation of seats to higher disproportionality.* This becomes apparent by a comparison of systems 3 and 4 in Table 3. Both of these systems display no systematic bias against small (large) parties; on average, all parties receive their exact PR proportion of seats. But while system 3 achieves this allocation precisely each time, system 4 does so only on average with slight deviations above and below that level. Yet, system 4 is accorded a higher level of disproportionality than system 3 according to the "deviations" based measures in Table 3b. It is trivial to see that the estimates of disproportionality from the procedure we propose reflect both the political character of disproportionality as well as the systematic or expected disproportionality, not its random component due to noisy deviations above or below a given expected value of seats represented in equation (4).

An instance when this drawback of deviation-based traditional indices becomes relevant is when comparing systems with or without compensatory "supplementary seats." The former are systematically treated as more proportional by traditional indices compared to – possibly – equally proportional systems that do not achieve such precise PR allocation. Of course, higher variation around some given level of (dis)proportional allocation is an additional potentially relevant dimension of the electoral system. Yet, such allocation variability should be distinguished from the systematic effects of the electoral institution in favor or against large (small) parties. The weak proportionality parameter δ (or α) used in this analysis is a better reflection of the latter, while indices calculated on the basis of "deviations" treat both allocation variability and disproportionality as the same thing.

Lastly, we point out that with the exception of the third Greek electoral system (GRE3), none of the analyzed systems displays levels of disproportionality consistent with the "cube law", *i.e.* weak (dis)proportionality α close to 3[25]. Indeed, among single member district plurality electoral systems, the Bristish one has a parameter close to $\alpha \simeq 1.56$ (UK2). We point out that similar estimates considerably below three have been obtained in other studies. For example, King, 1987, p. 171, using data from UK elections in the period 1950-1987 estimates $\alpha = 1.60$ for a two-party model incorporating partisan bias, while he obtains $\alpha = 1.14$ from a multiparty model also allowing for bias.

These results strongly suggest that disproportionality may be significantly influenced by additional provisions of the electoral law besides district magnitude. For example, UK's large number of district's and geographic distribution of party support enhance proportionality. On the other hand Greece's peculiar rules about allocation in upper tiers increase disproportionality (e.g. GRE 6, GRE7, GRE8, GRE10). On the contrary, the last two Austrian systems (AUT3, AUT4) are nearly perfectly proportional ($\delta = .5$, or $\alpha = 1$) due in part to upper tiers of allocation.

5. CONCLUSION

Essentially, measures of electoral disproportionality arise from the estimation of the parameters of some postulated low-dimensional representation of a generically multidimensional process. We argued that significant gains on the empirical fit of this representation are achieved if at least two parameters are included in this approximation: one to assess severe disproportionality in the form of electoral thresholds, and another to represent weak (dis)proportionality or responsiveness. Through the use of MCMC techniques we were able to simultaneously estimate these two quantities from actual electoral returns. The resultant two-dimensional summary of the electoral system is in many respects superior for the purposes of comparative empirical work.

Our threshold estimates have direct interpretation across electoral systems, since they all apply at the national – as opposed to district level – vote shares. Unlike imputations, these estimates take into account the effect of the pattern of distribution of partisan electoral

[25]Given the transformation $\delta = \dfrac{\alpha}{\alpha + 1}$ from equation (10), we have $\alpha = 3$ when $\delta = .75$.

forces across electoral districts. These measures also permit a direct separate assessment of the effect of nation-wide thresholds on the party system, separate from any effect of disproportionality for parties above thresholds. Our measure of the latter is a superior measure of the systematic component of disproportionality as well as of the "political character of disproportionality" compared to deviations-based indices. Lastly, unlike imputations or traditional indices, our procedure also results to a summary of the precision or confidence that can be placed on these parameters through the reported posterior standard errors.

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TABLE 1: Electoral Systems for National Parliaments in EU Countries, 1945-96

COUNTRY	SYSTEM	# OF ELECTIONS & YEARS
AUSTRIA	AUT1	7:1945-66
	AUT2	1:1970
	AUT3	6:1971-90
	AUT4	2:1994-95
BELGIUM	BEL1	17:1946-95
DENMARK	DEN1	4:1945-53
	DEN2	3:1953-60
	DEN3	14:1964-94
FINLAND	FIN1	15:1945-95
FRANCE	FRA1	3:1945-46
	FRA2	2:1951-56
	FRA3	2:1958-62
	FRA4	3:1967-73
	FRA5	4:1978-81,1988-93
	FRA6	1:1986
GERMANY	GER1	1:1949
	GER2	1:1953
	GER3	8:1957-83
	GER4	2:1987, 1994
	GER5	1:1990
GREECE	GRE1	2:1946-50
	GRE2	1:1951
	GRE3	2:1952-56
	GRE4	1:1958
	GRE5	3:1961-64
	GRE6	1:1974
	GRE7	2:1977-81
	GRE8	1:1985
	GRE9	3:1989-90
	GRE10	1:1993
IRELAND	IRE1	15:1948-92
ITALY	ITA1	1:1946
	ITA2	2:1948-53
	ITA3	9:1958-92
	ITA4	2:1994-96
LUXEMBOURG	LUX1	12:1945-1994
NETHERLANDS	NET1	3:1946-52
	NET2	12:1956-1992
PORTUGAL	POR1	9:1975-96
SPAIN	SPA1	7:1977-96
SWEDEN	SWE1	1:1948
	SWE2	6:1952-68
	SWE3	9:1970-94
UK	UK1	1:1945
	UK2	13:1950-92

Sources: *Mackie & Rose (1991, 1997), Lijphart (1994).*

TABLE 2: Posterior Moments (Restricted and Unrestricted Models)

SYSTEM	δ	StDev	τ	StDev	Effective Threshold†	$\delta\ (\tau_j = 0)$	StDev
AUT1	0.56	(0.013)	3.46	(1.82)	8.5	0.58	(0.012)
AUT2	0.56	(0.040)	2.95	(1.52)	2.6‡	0.58	(0.034)
AUT3	0.50	(0.015)	3.25	(1.23)	2.6‡	0.53	(0.011)
AUT4	0.50	(0.023)	3.09	(1.36)	--	0.53	(0.019)
BEL1	0.54	(0.006)	1.25	(1.28)	4.8	0.55	(0.005)
DEN1	0.51	(0.015)	2.18	(1.31)	1.6	0.52	(0.014)
DEN2	0.52	(0.012)	0.20	(0.12)	2.6	0.52	(0.013)
DEN3	0.50	(0.007)	2.03	(1.25)	2*	0.53	(0.006)
FIN1	0.54	(0.007)	1.45	(1.37)	5.4	0.55	(0.007)
FRA1	0.57	(0.014)	2.35	(3.12)	12.9	0.57	(0.014)
FRA2	0.48	(0.018)	2.60	(3.12)	12.7	0.47	(0.020)
FRA3	0.56	(0.013)	3.34	(2.80)	35	0.58	(0.011)
FRA4	0.56	(0.009)	1.14	(0.49)	35	0.56	(0.008)
FRA5	0.60	(0.008)	1.88	(1.38)	35	0.63	(0.007)
FRA6	0.54	(0.014)	0.98	(0.13)	11.7	0.56	(0.012)
GER1	0.52	(0.013)	0.16	(0.09)	5	0.52	(0.013)
GER2	0.55	(0.011)	0.41	(0.22)	5	0.55	(0.011)
GER3	0.51	(0.007)	4.78	(2.63)	5	0.55	(0.006)
GER4	0.50	(0.011)	3.77	(1.76)	5	0.53	(0.009)
GER5	0.54	(0.013)	2.27	(0.10)	5	0.58	(0.010)
GRE1	0.54	(0.013)	1.54	(0.69)	--	0.54	(0.013)
GRE2	0.63	(0.018)	0.61	(0.35)	--	0.63	(0.018)
GRE3	0.76	(0.018)	1.67	(1.49)	--	0.76	(0.018)
GRE4	0.65	(0.018)	1.48	(0.85)	--	0.65	(0.018)
GRE5	0.60	(0.014)	5.04	(3.94)	--	0.60	(0.014)
GRE6	0.63	(0.015)	4.88	(2.66)	18.8	0.63	(0.014)
GRE7	0.63	(0.012)	3.25	(3.41)	16.1	0.64	(0.011)
GRE8	0.63	(0.024)	0.95	(0.52)	14.7	0.63	(0.023)
GRE9	0.56	(0.013)	0.33	(0.18)	--	0.57	(0.012)
GRE10	0.55	(0.022)	3.67	(0.60)	3*	0.58	(0.017)
IRE1	0.53	(0.007)	1.11	(1.10)	17.2	0.54	(0.007)
ITA1	0.51	(0.012)	0.17	(0.10)	0.1	0.51	(0.012)
ITA2	0.53	(0.007)	0.18	(0.12)	2.4	0.53	(0.007)
ITA3	0.53	(0.003)	0.11	(0.12)	2	0.53	(0.003)
ITA4	0.57	(0.011)	0.09	(0.05)	--	0.57	(0.010)
LUX1	0.53	(0.016)	3.14	(2.64)	5.1	0.54	(0.015)
NET1	0.51	(0.021)	1.01	(0.63)	1*	0.51	(0.020)
NET2	0.51	(0.007)	0.62	(0.41)	0.67*	0.52	(0.006)
POR1	0.56	(0.008)	2.72	(2.73)	5.7	0.58	(0.007)
SPA1	0.56	(0.005)	0.18	(0.12)	10.2	0.56	(0.004)
SWE1	0.53	(0.022)	3.17	(1.83)	8.5	0.53	(0.022)
SWE2	0.52	(0.009)	1.52	(1.37)	8.4	0.52	(0.009)
SWE3	0.50	(0.006)	3.47	(1.04)	4	0.52	(0.006)
UK1	0.59	(0.014)	0.10	(0.05)	35	0.59	(0.014)
UK2	0.61	(0.005)	0.63	(1.71)	35	0.61	(0.005)

Sources: Calculated by the author. Effective Threshold is from Lijphart (1994).
‡ Systems merged in Lijphart (1994).
* System-wide known threshold.

TABLE 3: Performance of Disproportionality Indices

(a) Alternative Allocation Patterns

PARTY	VOTE SHARE	ALLOCATION OF SEATS ACCORDING TO SYSTEM:					
		1	2	3	4†	5	6
A	40%	45%	35%	40%	$40 \pm p$ %	53%	63%
B	35%	35%	35%	35%	$35 \pm p$ %	47%	32%
C	25%	20%	30%	25%	$25 \pm p$ %	0%	5%

(b) Measured Disproportionality

INDEX	DISPROPORTIONALITY OF SYSTEM:					
	1	2	3	4	5	6
Rae*	3.33	3.33	0	$2p/3$	16.66	15.
Loosemore-Handby*	5	5	0	p	25	23
Gallagher*	5	5	0	p	2.17	2.17

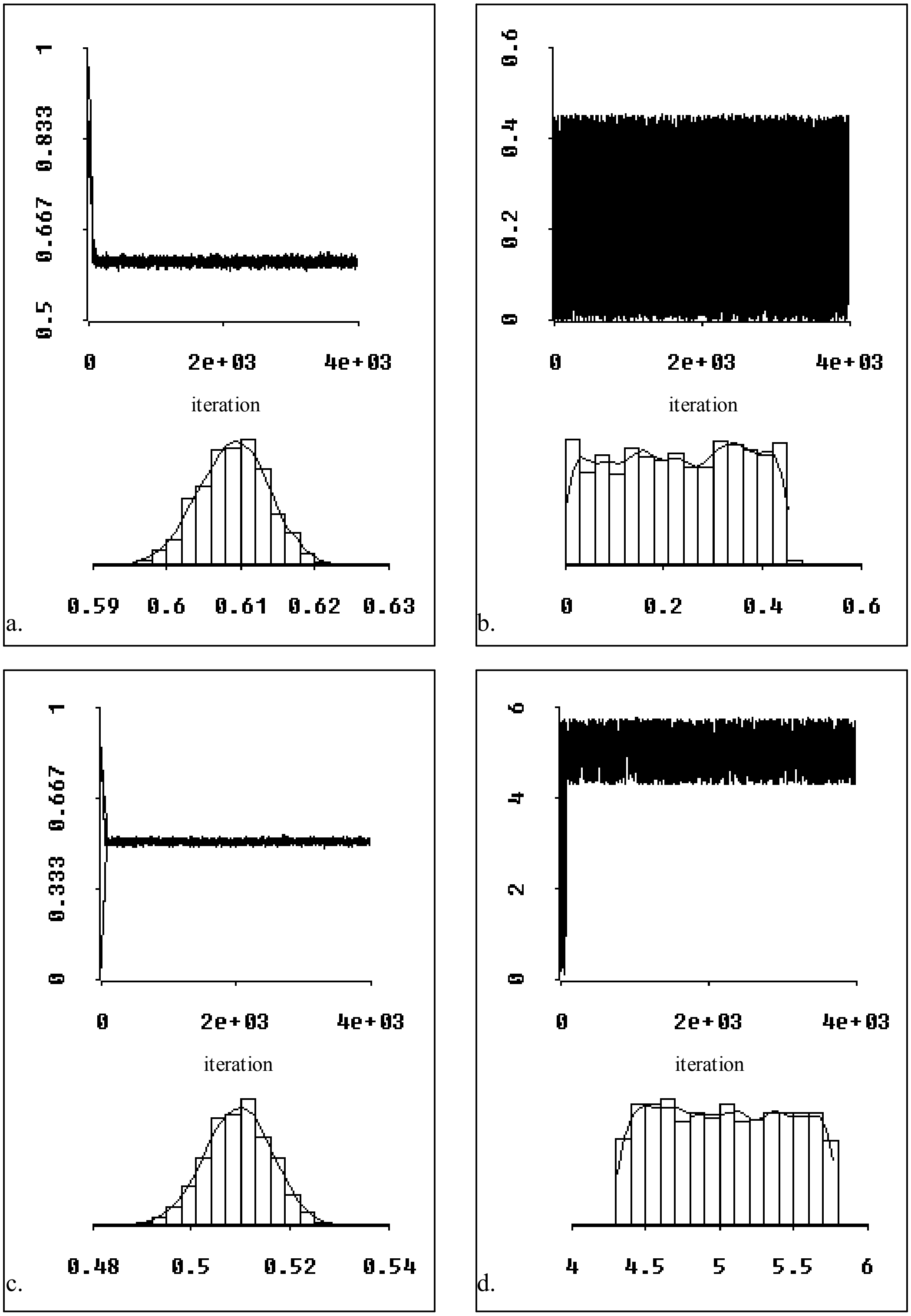
Sources: *Hypothetical data provided by the author.*

† Parties A, B, and C receive (40+p, 35, 25-p), (40-p, 35+p, 25), or (40, 35-p, 25+p) respectively with probability one-third each..

*Rae= $\frac{1}{n}\sum_{i=1}^{n}\left|v_i - s_i\right|$, Loosemore-Handby= $\frac{1}{2}\sum_{i=1}^{n}\left|v_i - s_i\right|$, Gallagher= $\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(v_i - s_i\right)^2}$.

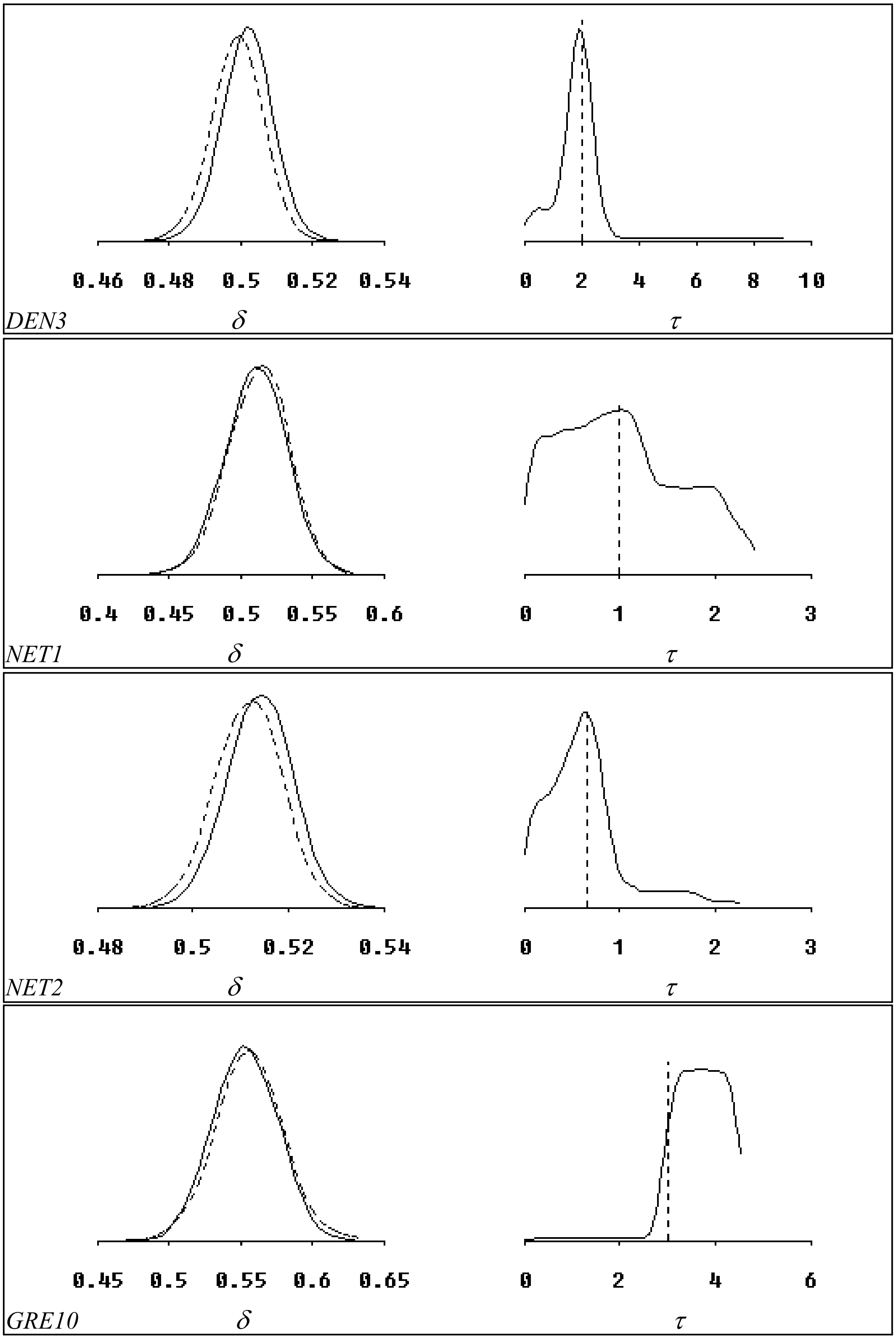
Key: *SYSTEM 1: Disproportional allocation, SYSTEM 2: Reverse disproportionality, SYSTEM 3: "Exact" PR, SYSTEM 4: "Noisy" PR, SYSTEM 5: PR with electoral threshold between 25% and 35%, and SYSTEM 6: Highly disproportional allocation.*

FIGURE 1: Gibbs Chains and Posterior Distributions of Selected Parameters



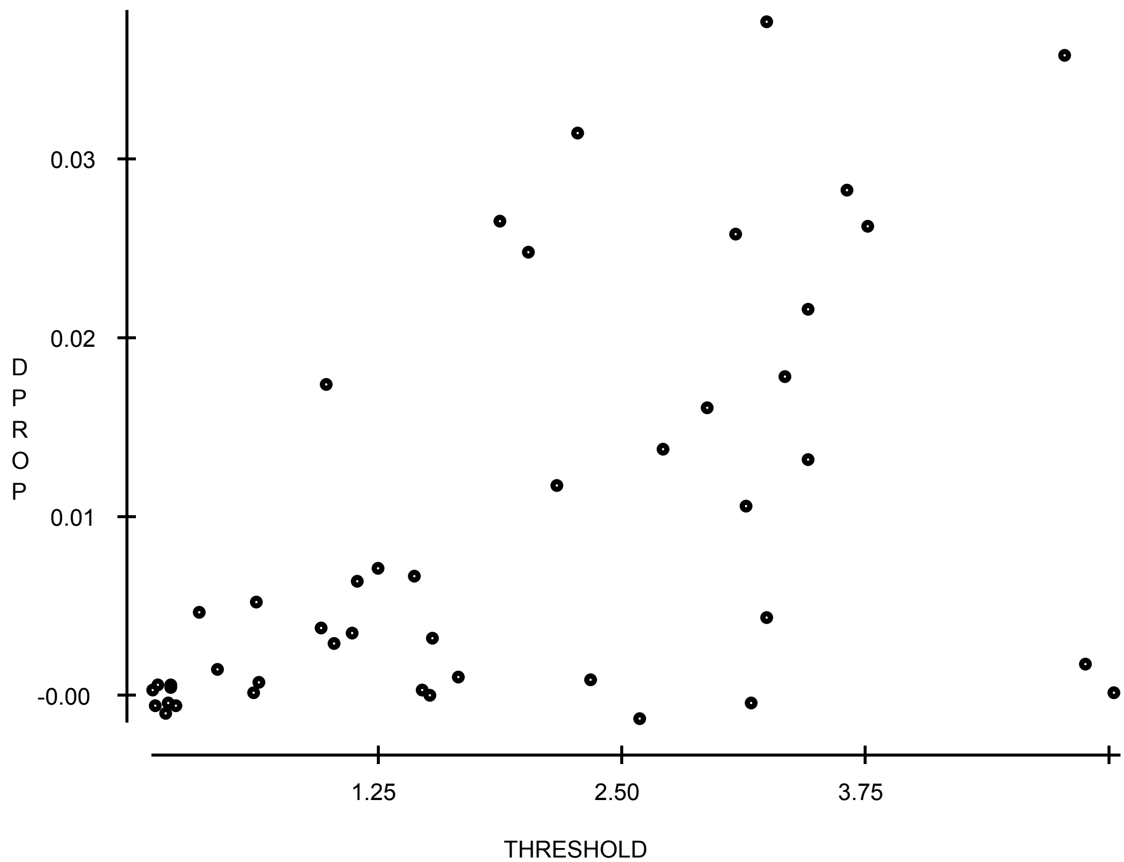
Key: (a). *UK2:* δ, (b). *UK2:* τ_j, *1950 election,* (c). *GER3:* δ, (d). *GER3:* τ_j, *1969 election.*
Number of chains: 3. Number of iterations: 4000. Posteriors based on last 3500 variates of each chain.

FIGURE 2: Posterior Distributions (Threshold is Known)



Key: ——— *Threshold Assumed Unknown,* - - - - - - - - *Known Threshold.*

FIGURE 3: Overestimation of Weak Proportionality under the Restricted Model



Sources: *Table 2.*

key

DPROP = Difference between estimated restricted and unrestricted estimates of weak proportionality, $\delta\left(\tau_j = 0\right) - \delta$.

THRESHOLD = Estimated thresholds τ.