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Dynamics of Majority Rule With Endogenous Reversion Point: The Distributive Case[1]

by

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Abstract: We analyze a divide-the-dollar majority rule bargaining game with an endogenous reversion point. A new dollar is divided in each of an infinity of periods according to the proposal of a probabilistically recognized legislator – if a majority prefer so – or according to previous period's decision otherwise. Players' payoff is the discounted sum of per period utilities. We show existence of a Markov Perfect Nash Equilibrium; it is such that irrespective of initial conditions outcomes are absorbed within an irreducible, finite set consisting of allocations that give the entire dollar to the proposer. Contrary to results for amendment agendas without future interaction after the final vote, the decision within each period may be covered by the status quo. Outcomes may be *ex ante* Pareto inefficient. For sufficiently large legislature a dictatorial agenda setter can impose her ideal point in all periods but the initial two a la McKelvey, 1976, 1979, even though legislators are farsighted. The equilibrium collapses for high degrees of risk aversion, or when the legislature is small and recognition probabilities are asymmetric. Contrary to the comparative static in the Baron-Ferejohn model (Eraslan, 2002) higher probabilities of recognition render legislators less expensive coalition partners *ceteris paribus.*

Keywords: Endogenous Reversion Point, Dictatorial Agenda Setting Power, Legislative Bargaining, Markov Perfect Nash Equilibrium, Stage Undominated Voting strategies, Uncovered Set.

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1. INTRODUCTION

In modern democracies the legislative power that permits the enactment of particular policies, most often also allows their revision at future points in time. Legislative interaction is in that sense truly dynamic: not only must legislators consider the immediate consequences of legislation they enact, but they must also anticipate that this decision will serve as the alternative (*status quo*) which they have to compare with potential future decisions. Given the results of Romer and Rosenthal, 1978, replicated in numerous models of legislatures regarding the influence of the status quo on legislative outcomes and the power of the proposer, there is ample theoretical support for the claim that considerations about future consequences of current decisions should be significant in the calculus of legislators.

Yet, it is a study of the intra-temporal dynamics involved in reaching a single decision that predominate in most theoretical studies of legislatures. With few exceptions (Epple and Riordan, 1987, Baron, 1996, Dixit, Grossman, and Gul, 2000, and Kalandrakis, 2002[1]) legislative studies implicitly or explicitly assume that the legislature adjourns after a decision is reached, or that the legislative jurisdiction is such that no future revision is possible or probable after a finite number of periods.

This neglect for the inter-temporal dynamics of majority rule choice is also unfortunate because it is in this context that the cycling results of social choice theory become relevant. Since generically (Plott, 1967, Schofield, 1978, 1983, Rubinstein, 1979) no policy beats every other by majority rule and all alternatives are entangled in a big cycle (McKelvey, 1976, 1979), it is not only possible that legislatures revise policies too often, but also that they do

[1]See also Banks and Duggan, 2001, for a dynamic model where the status quo may prevail for any number of periods until a proposal is approved.

so in a way that makes future policies completely indeterminate[2]. But even if *chaos* does not prevail, McKelvey, 1976, 1979, shows that in such environments a dictatorial agenda-setter can force any decision upon the legislature if legislators are myopic by appropriately constructing a sequence of pair-wise comparisons.

These predictions are potentially reversed if legislators are strategic and face a sequence of votes on alternatives (an agenda) endogenously or exogenously constructed. These intra-temporal dynamics are studied for example by Shepsle and Weingast, 1984, Banks, 1985, and McKelvey, 1986, in the context of amendment agendas. With an amendment agenda, no future interaction after the final vote, and sophisticated voting (Farquharson, 1969), possible final outcomes belong in the uncovered set (Fishburn, 1979, Miller, 1980). The endogenous construction of amendment agendas is considered by Banks and Gamsi, 1986, Austen-Smith, 1987, Duggan, 2001, and Penn, 2001.

Results for amendment agendas do not generalize when considering other agenda institutions. For example, Bernheim, Rangel, and Rayo, 2002, show that with sufficient periods or proposers, endogenously constructed agendas lead with high probability to an extreme outcome. In their analysis, though, the agenda is such that intermediate voting rounds displace the existing status quo while legislators only care about the final victor in the sequence of decisions. Since constitutionally status quo legislation is displaced only if a new law is promulgated, the agenda they consider is applicable in the limited range of cases

[2]There is another interpretation of the chaos theorems that is antithetical to the one we discuss and focuses on the intra-period or intra-temporal dynamics of reaching a decision: for any pending decision an entrepreneurial minority can always find an alternative that is majority preferred and, as a result, legislatures are *unable to decide*. We consider this interpretation rather irrelevant. Even if such intra-temporal dynamics prevail legislatures do decide in these cases, *i.e.* they decide to preserve the status quo.

when legislation promulgated after each voting stage provides for implementation of policies at the future point in time when the final round of voting takes place. While outcomes in their analysis fall in the Pareto set, Ordeshook and Scwartz, 1986 show with considerable generality that there exist (exogenously constructed) agendas that implement *any* outcome despite sophisticated voting.

Our goal in this paper is to study the dynamics of the strategic interaction of legislators who can enact legislation and subsequently revise their decision in each of an infinity of periods. In such a dynamic framework incentives for moderation arise both on the part of the voters via sophisticated voting, as well as on the part of proposers. The intuition follows directly from the results of Romer and Rosenthal, 1978, regarding the power of the proposer as a function of the status quo. In a nutshell, by adopting an extreme (but desirable) policy in the current period, a proposer faces the risk that future proposers will achieve passage of undesirable policies because they will legislate with a more distant status quo. Thus, there exists a trade-off between the immediate utility arising from the current period legislative decision and the stream of payoffs following this decision.

This intuition does seem to be at work in related studies where policy decisions are drawn from ideological spaces. Baron, 1996, studies a model with the same institutions as in this study where decisions are drawn from a one-dimensional space of alternatives. He shows that legislative outcomes converge to the median from arbitrary initial policy decision and discusses the calculation of proposers that may strategically place the status quo in order to avoid undesirable future policies. Baron and Herron, 1999, numerically analyze a finitely repeated version of the same game with a two-dimensional policy space and three legislators with Euclidean stage preferences. They find that equilibrium legislative decisions tend to be more centrally located with a higher discount factor and a longer time horizon.

We show that incentives for moderation are considerably weaker in a distributive policy space. We study these incentives in a dynamic game where the legislative decision in each period is the division of a new dollar. Players only care about the share of funds they receive and their payoff is the discounted sum of per period utilities. In each period, one of the players is recognized to make a proposal for the division of the dollar. If the proposal obtains a majority, then the division is implemented, else the dollar is divided as it was in the last period. We allow general, asymmetric probabilities of recognition and risk-aversion[3].

Although no general existence results apply for this class of dynamic games, we show that a 'simple' refined equilibrium exists for low enough degrees of risk-aversion. It is simple in that legislators condition their behavior only on the status quo in each period of deliberation and not on the whole history of the game. Despite the existence of majority rule cycling and the fact that different legislators may propose alternatives in any given period, only a finite number of budgets are eventually voted and decisions are identical between any two consecutive periods with positive probability. Thus we observe no chaos, nor perpetual instability.

Yet, the equilibrium involves extreme allocations. Within at most three periods from the beginning of the game and irrespective of the initial status quo the proposer receives the whole dollar and that pattern persists ever after. Although within any given period the agenda we consider is a (degenerate) amendment agenda, outcomes under this equilibrium

[3]The model falls in the tradition of sequential models of bargaining as in Rubinstein, 1982, and Baron and Ferejohn, 1989. Kalandrakis, 2002, analyzes the same game as in this study among three legislators with equal probabilities of recognition and under the assumption of risk neutrality of stage preferences.

fall outside the uncovered set and may be covered by the status quo when the covering relation is defined on the basis of stage preferences.

We also show that for any discount factor there exists a sufficiently large legislature in which a dictatorial agenda-setter that is recognized with probability one extracts her ideal budget in each period (as in McKelvey 1976) even though legislators (voters) are farsighted. The differences between our findings and those in Baron, 1996, and Baron and Herron, 1999, for ideological spaces may account for the common institutional choice that legislation on distributive policy spaces is often deliberated under special institutional rules compared to the rules applicable for ideological legislation.

The equilibrium collapses if either the degree of risk-aversion is high relative to the size of the legislature or – for fixed levels of risk-aversion – if the legislature is small. Thus, incentives for nasty distributive politics diminish when the stage utility of legislators displays high levels of risk aversion or rises steeply, so that the additional gain from reducing the current allocation of other players diminishes. Similarly, these incentives increase with larger legislatures or when a larger number of recipient units are involved in the allocation of transferable resources. These results suggest that it may be optimal to increase the level of geographical aggregation at which budgets are divided. Similarly, federal or confederate structures may experience increased incentives for conflict-ridden distributive politics with excessive expansion.

The finding about the effect of risk aversion is the opposite from the model of Baron and Ferejohn, where risk aversion allows the proposer to extract more of the surplus (Harrington, 1990). Also contrary to the comparative statics in the Baron Ferejohn model (Eraslan, 2002) legislators that have high probability of being the proposer are less expensive coalition partners, *ceteris paribus*. By accepting an onerous budget today, legislators are able to extract more of the budget tomorrow. Thus legislators with a high probability of being the proposer in the next period are willing to accept proposals that allocate them a small share of the budget.

In what follows we present the model in detail and define the equilibrium solution concept. We offer a description of the equilibrium along with a characterization of equilibrium strategies for a subset of possible status quo in section 3. In section 4 we state and discuss the main findings. We conclude in section 5.

2. LEGISLATIVE SETUP & EQUILIBRIUM NOTION

Consider a set $N = \{1, ..., n\}$ of $n = 2\kappa + 1$ committee members, $\kappa \geq 2$, that convene in each of periods $t = 1, 2, ...$ to choose a legislative outcome $\mathbf{x}^t \in \Delta$. The space of possible decisions in each period, Δ, represents possible divisions of a fixed budget (a dollar), *i.e.* $\Delta \equiv \{\mathbf{x} \in \mathbf{R}^n : x_i \geq 0, i \in N, \sum_{i=1}^n x_i = 1\}$. At the beginning of period $t = 1, 2, ...$ legislator i is recognized with probability $\pi_i \geq 0$, $i \in N$, $\sum_{i=1}^n \pi_i = 1$, to make a proposal $\mathbf{z} \in \Delta$. Having observed the proposal legislators vote *yes* or *no*. If $m = \kappa + 1$ or more vote *yes* then the proposal is implemented in that period, *i.e.* $\mathbf{x}^t = \mathbf{z}$. Otherwise, the dollar is split as it was in the previous period, *i.e.* $\mathbf{x}^t = \mathbf{x}^{t-1}$. Thus, previous period's decision $\mathbf{x}^{t-1}$ serves as the *status quo* or *reversion point* in the current period t, with the initial reversion point, $\mathbf{x}^0 \in \Delta$, exogenously given.

Legislators derive vNM stage utility $u_i : \Delta \longrightarrow \mathbf{R}$, $i \in N$, from the implemented proposal $\mathbf{x}^t$. We assume that $u_i(\mathbf{x}) = u(x_i)$, for all $i \in N$ with u continuous, $u' > 0$, $u'' \leq 0$, $u(0) = 0$, and $u(1) = 1$. In the special case $u'' = 0$, we obtain $u_i(\mathbf{x}) = x_i$, *i.e.* legislators are risk-neutral as in Kalandrakis, 2002. We will generally admit risk-aversion that opens the possibility of inefficient outcomes, and add more restrictive conditions as necessary. The future is discounted by a common factor $\delta \in [0,1)$, so that the utility of legislator i from a sequence of legislative outcomes $\{\mathbf{x}^t\}_{t=1}^{+\infty}$ is given by:

$$V_i\left(\left\{\mathbf{x}^t\right\}_{t=1}^{+\infty}\right) = \sum_{t=1}^{+\infty} \delta^{t-1} u_i\left(\mathbf{x}^t\right) = \sum_{t=1}^{+\infty} \delta^{t-1} u\left(x_i^t\right), i \in N \tag{1}$$

If $\mathbf{z}_i^t \in \Delta$ denotes the (observed) proposal of player i when recognized in period t, and $\mathbf{v}^t \in \{yes, no\}^n$ is the vector of voting decisions in the same period, a *history* $h_v^t \in H_v^t$ at the voting stage of period t is a vector $\left(\mathbf{x}^0, \mathbf{z}_j^1, \mathbf{v}^1, ..., \mathbf{z}_h^{t-1}, \mathbf{v}^{t-1}, \mathbf{z}_g^t\right)$, where legislators j, h, and g where recognized in periods 1, $t-1$, and t respectively. Likewise, a history $h_p^t \in H_p^t$ at the proposal stage of period t is a vector $\left(\mathbf{x}^0, \mathbf{z}_j^1, \mathbf{v}^1, ..., \mathbf{z}_h^{t-1}, \mathbf{v}^{t-1}\right)$. Strategies in this game are sequences of functions that map *histories* to the space of proposals and voting decisions. Pure proposal strategies for player i are determined by a sequence of functions $f_{p,i}^t : H_p^t \rightarrow \Delta$, and voting strategies by a sequence of functions $f_{v,i}^t : H_v^t \rightarrow \{yes, no\}$, $t = 1, 2, ...$

In what follows, though, we restrict analysis to cases when players condition their behavior only on a summary of the history of the game that accounts for *payoff-relevant* effects of past behavior (Maskin and Tirole, 2001, Fudenberg and Tirole, ch. 13). Specifically, let the *state* $\mathbf{s} \in S$ in period t be defined by previous period's allocation, *i.e.* $\mathbf{s} = \mathbf{x}^{t-1}, S = \Delta$. Denote the space of Borel probability measures over Δ by $\wp(\Delta)$. A (mixed) *proposal strategy* for legislator i conditional on the state $\mathbf{s}$, $\mu_i[\mathbf{s}] \in \wp(\Delta)$, represents a probability measure over legislative outcomes proposed by legislator i when recognized with status quo $\mathbf{s}$. A *voting strategy* conditional on the state $\mathbf{s}$ is a Borel measurable *acceptance set* $A_i(\mathbf{s}) \equiv \{\mathbf{z} \in \Delta \mid i \text{ votes } yes \text{ if state is } \mathbf{s}\}$ for legislator i over proposals $\mathbf{z}$. Thus, a (mixed) *Markov strategy* for legislator i is a function from the state space S to the space of proposal and voting strategies. We will denote such strategies by $\sigma_i(\mathbf{s}) = (\mu_i[\mathbf{s}], A_i(\mathbf{s}))$. Restricting[4] analysis to Markov strategies amounts to the requirement that players behave identically in different periods with the same state, even if that state arises from different histories. We believe there are at least two sets of reasons that make it theoretically important to study equilibrium behavior under these assumptions.

One set of arguments revolves around certain appealing features of the Markovian behavioral assumption. Markov strategies are more likely to be followed by players in complex strategic environments like the one we analyze because of their simplicity. As Maskin and Tirole, 2001, page 193, argue, Markov strategies "*...prescribe the simplest form of behavior that is consistent with rationality.*" In the same spirit and a related model, Baron and Kalai, 1993, show that the Markovian (in their case stationary) equilibrium analyzed in the model of Baron and Ferejohn, 1989, is the unique least complex (or simplest) subgame perfect equilibrium where simplicity is defined as the number of different states required to specify the strategy of an automaton[5]. Second, Markov strategies imply that players take to heart the idea that *bygones are bygones* a motivating concept behind the notion of subgame perfection (Maskin and Tirole, 2001). Thirdly, Markov equilibria imply that *small* causes have *small* effects (Maskin and Tirole, 2001).

There are also considerable analytical gains in focusing on Markov equilibria. Non-Markovian equilibria for games of the type we analyze proliferate to the extent that equilibrium outcome becomes completely indeterminate[6]. Second, by focusing on equilibria where

[4] If players play Markov strategies the restriction to Markovian best responses is consistent with equilibrium.

[5] Obviously this is not the only possible definition of simplicity, and others may be more appropriate for the game we analyze. We leave such issues for future research.

[6] We do note that folk theorems for stochastic games assume a combination of finite action and state

behavior is only conditioned on the status quo, we allow for crisper analysis of the effect of this variable on policy outcomes. Third, our results are comparable to those in Baron, 1996, and Baron and Herron, 1999, who analyze the same institutions in a different policy space than ours and use the same solution concept.

To state the solution concept rigorously, we need additional notation. Define the *winset* of $\mathbf{x} \in \Delta$ as:

$$W(\mathbf{x}) = \left\{ \mathbf{y} \in \Delta \mid \sum_{i=1}^{n} I_{A_i(\mathbf{x})}(\mathbf{y}) \geq m \right\}. \tag{2}$$

The winset contains the alternatives in Δ that are majority preferred to the status quo $\mathbf{s}$ according to Markov voting strategies $A_i(\mathbf{s})$. Then, given a n-tuple of Markov strategies $\sigma = \{\sigma_i\}_{i=1}^{n}$, we can recursively define the *continuation value*, $v_i(\mathbf{s})$, of legislator i when the state is $\mathbf{s}$ as:

$$v_i(\mathbf{s}) = \int_{\Delta} [u_i(\mathbf{x}) + \delta v_i(\mathbf{x})] \, Q[d\mathbf{x} \mid \mathbf{s}] \tag{3}$$

where for any measurable $Y \subseteq \Delta$, $Q[Y \mid \mathbf{s}]$ represents transition probabilities defined as

$$Q[Y \mid \mathbf{s}] \equiv \sum_{i=1}^{n} \pi_i \mu_i [Y \cap W(\mathbf{s}) \mid \mathbf{s}] + I_Y(\mathbf{s}) \sum_{i=1}^{n} \pi_i \mu_i [\Delta \diagdown W(\mathbf{s}) \mid \mathbf{s}] \tag{4}$$

The first part of equation (4) accounts for transitions to proposals that obtain a majority, while the second part represents transitions to the *reversion point* or *status quo* $\mathbf{s}$ when the

spaces, public randomization, and/or impose conditions on feasible transitions (e.g. Dutta, 1995). Levine, 2000, shows that the folk theorem fails in a robust way for a dynamic game that does not meet Dutta's transition conditions. Yet, these conditions are satisfied in our game; and in a related but not identical bargaining game with three players, Epple and Riordan, 1987, show at least two radically different outcomes can be sustained as subgame perfect equilibria. Thus, we conjecture that all individually rational long-run payoffs can be supported as subgame perfect equilibria of the game we analyze.

proposal does not receive a majority. On the basis of equation (3) define the expected utility of legislator i as a function of the current decision $\mathbf{x}^t$:

$$U_i\left(\mathbf{x}^t\right) = u_i\left(\mathbf{x}^t\right) + \delta v_i\left(\mathbf{x}^t\right) \tag{5}$$

where it is understood that $W(\mathbf{x})$, $Q[Y \mid \mathbf{s}]$, $v_i(\mathbf{x}^t)$ – hence $U_i(\mathbf{x}^t)$ – are defined for given Markov strategies σ. Then:

Definition 1 *A Markov Perfect Nash Equilibrium in Stage-Undominated Voting strategies (MPNESUV) is a set of Markov strategies* $\sigma^* = \{\sigma_i^*\}_{i=1}^n = \{(\mu_i^*[\mathbf{s}],\ A_i^*(\mathbf{s}))\}_{i=1}^n$, *such that for all* $i \in N$, *and for all* $\mathbf{s} \in S$*:*

$$\mathbf{y} \in A_i^*(\mathbf{s}) \Longleftrightarrow U_i(\mathbf{y}) \geq U_i(\mathbf{s}) \tag{6}$$

$$\mu_i^*\left[\arg\max\left\{U_i(\mathbf{x}) \mid \mathbf{x} \in W(\mathbf{s})\right\} \mid \mathbf{s}\right] = 1 \tag{7}$$

The first equilibrium condition amounts to the requirement that players use stage-undominated (Baron and Kalai, 1993) voting strategies, *i.e.* they only vote *yes* to proposals they weakly prefer over the status quo $\mathbf{s}$. Thus, we eliminate a – rather large – class of equilibria that involve players approving proposals with more than a bare majority of *yes* votes while a majority prefer the status quo. Such equilibria solely rely on the fact that changes in individual votes do not alter the voting outcome, so that both voting *yes* or *no* constitute a best response for all players. The second equilibrium condition requires that committee members optimize when making proposals. Note that we also require that proposers always propose alternatives that obtain a majority; since equilibrium condition (6) ensures that $\mathbf{s} \in W(\mathbf{s}) \neq \emptyset$, this requirement and equilibrium condition (7) are consistent.

3. EQUILIBRIUM ANALYSIS

We have specified a dynamic game with continuous action and state spaces, and no existence theorem for Markov equilibria is applicable in this case[7]. We will establish existence of equilibrium by construction. Our approach consists of the following steps: first, we conjecture and characterize equilibrium strategies for a subset of the space of possible decisions. We conjecture that this subset constitutes an absorbing set and derive several results for the expected utility function of legislators for outcomes within that set. We then establish the validity of the conjecture by showing that there exist optimal strategies for all status quo outside this absorbing set that move the game within this set in a single period. Proposal and voting strategies for status quo both within and outside the absorbing set satisfy equilibrium conditions (6) and (7), hence we obtain a MPNESUV.

Additional notation will be necessary in order to elaborate on the nature of the equilibrium conjecture. Partition the space of policy outcomes into subsets $\Delta_\theta \subset \Delta$, where $0 \leq \theta < n$ indicates the number of legislators receiving zero share of the dollar:

$$\Delta_\theta = \left\{ \mathbf{x} \in \Delta \mid \sum_{i=1}^{n} I_{\{0\}}(x_i) = \theta \right\} \tag{8}$$

Our conjecture is built on the intuition that equilibrium proposals involve 'minimum winning coalitions' (Riker, 1962), such that at most $m = \kappa + 1$ legislators receive a positive fraction of the dollar in each period. As a result, Δ_θ with $\theta \geq \kappa$ is an absorbing set, one that is reached in at most one period from any initial allocation of the dollar.

[7]Indeed (subgame perfect) equilibirum may fail to exist in such games (Harris, Reny, and Robson, 1995). Existence theorems apply only in special classes of stochastic games with uncountable action and state spaces (e.g. Chakrabarti, 1999). Generalized equilibria that require some form of public randomization also exist (e.g. Duffie et al. 1994).

Now suppose exactly a majority of legislators have a positive fraction of the dollar, *i.e.* $\mathbf{s} \in \Delta_\kappa$. If legislators with zero share of the dollar are willing to accept a different (optimal) proposal that allocates them zero as well[8], then the proposer can extract the whole dollar if the status quo allocates her a positive amount. If the proposer's allocation is zero instead, she may need to allocate a positive amount to one more player in order to obtain a majority for her proposal. In either case, starting from an allocation in Δ_κ, the game moves into set $\Delta_{\theta>\kappa}$ in one period since $n \geq 5$. Once this happens there exists a sufficient number of legislators with zero share of the dollar in order for the proposer to extract the whole dollar ever after.

<<INSERT TABLE 1 ABOUT HERE>>

We illustrate this path of play with an example in Table 1 for the case $n = 5$. The equilibrium conjecture suggests the possibility of solving the game backwards from the period when absorption to the set of outcomes that give zero to $n-1 = 2\kappa$ legislators takes place, to arbitrary initial allocation of the dollar. It is by means of this strategy that we demonstrate the advertised result.

In the remainder of this section we characterize equilibrium proposals and continuation values for all cases when the *status quo* is an allocation with a bare minority κ or more legislators having zero share of the dollar (*i.e.* θ is equal to $n-1$, $n-2$, ..., m, κ, respectively). We derive continuation values on the basis of these proposal – and voting – strategies. We then use the expected utility function of players derived from this construction in the following section where we establish that these proposal and voting strategies form part of

[8]As we shall show, legislators strictly prefer such proposals in this case because they can extract more of the dollar in the next period.

a MPNESUV.

i. $\kappa < \theta \leq 2\kappa = n-1$ Denote elements of $\Delta_{2\kappa}$ by $\mathbf{e}^j$ where $e_j^j = 1$, *i.e.* the dollar is allocated to player j. According to the conjectured equilibrium for all $\mathbf{s} \in \Delta_\theta$ with $\kappa < \theta \leq 2\kappa$ the proposer can form a majority to extract the whole dollar. Since for all subsequent periods we have $\mathbf{s} \in \Delta_{2\kappa}$, the proposer can successfully extract the whole dollar ever after. Hence, in any period with $\mathbf{s} \in \Delta_\theta$, $\kappa < \theta \leq 2\kappa$, each $i \in N$ expects to receive utility $u(1) = 1$ with probability π_i and utility $u(0) = 0$ otherwise in all subsequent periods. We can then write the continuation value of player i as:

$$v_i(\mathbf{s}) = \overline{v}_i = \frac{\pi_i}{1-\delta}, i \in N, \mathbf{s} \in \Delta_{\theta>\kappa} \tag{9}$$

ii. $\theta = \kappa$ Unlike the above cases, when $\theta = \kappa$ some proposers cannot secure a majority in order to extract the whole dollar. This occurs when the proposer is one of the κ legislators with zero share of the dollar. Without loss of generality, let $\mathbf{s}$ be such that $0 = s_1 = s_2 = ... = s_\kappa < s_m \leq s_{m+1} \leq ... \leq s_n$. This situation is depicted graphically in Figure 1.

<<INSERT FIGURE 1 ABOUT HERE>>

If legislator $i < m$ is recognized, she must allocate a positive amount to (at least) one of legislators m,...,n. Suppose she does so for only one of those legislators, say legislator j. Call this proposal $\mathbf{z} \in \Delta_{n-2}$. Since $n-2 > \kappa$, $\mathbf{z}$ is such that legislators' continuation value is determined by equation (9). Then, the proposer's utility from $\mathbf{z}$ is $U_i(\mathbf{z}) = u_i(\mathbf{z}) + \delta v_i(\mathbf{z}) = u(z_i) + \delta\overline{v}_i$, *i.e.* the proposer wishes to minimize the amount $z_j = 1 - z_i$ allocated to coalition partner j.

In general, this amount is a function of the state $\mathbf{s}$, so denote it by $z(\mathbf{s})$. If all $i < m$ play a pure proposal strategy (we shall soon show this is not always the case in equilibrium) and choose legislator j as coalition partner, then $z(\mathbf{s})$ must be such that $u(z(\mathbf{s})) + \delta\overline{v}_j = u(s_j) + \delta\left[\pi_j u(1) + \sum_{i=1}^{\kappa} \pi_i u(z(\mathbf{s})) + \left(1 - \sum_{i=1}^{\kappa} \pi_i - \pi_j\right) u(0) + \delta\overline{v}_j\right]$. From this we obtain $u(z(\mathbf{s})) = \dfrac{u(s_j)}{1 - \delta\sum_{i=1}^{\kappa} \pi_i}$. $z(\mathbf{s})$ is minimum when $j = m$ and we can assume under this conjectured play that the proposer chooses the legislator with the smallest positive share of the dollar s_m.

We will now show that these proposal strategies may not be equilibrium depending on the distribution of s_j, $j \geq m$. To this end, define the *demand* of a legislator as follows:

Definition 2 *The demand of legislator i for an alternative $\mathbf{x}$, $d_i(\mathbf{x})$ is*

$$d_i(\mathbf{x}) \equiv \max\{U_i(\mathbf{x}) - \delta\overline{v}_i, 0\} \tag{10}$$

The demand $d_i(\mathbf{x})$ is the minimum level of stage utility required by player i in order to vote *yes* on a proposal in $\Delta_{\theta>\kappa}$ when the status quo is $\mathbf{x}$. If legislator m is always chosen as coalition partner as above, we have $d_m(\mathbf{s}) = \dfrac{u(s_m)}{1 - \delta\sum_{i=1}^{\kappa} \pi_i}$. The demand of legislator $h > m$ is obtained by solving $d_h(\mathbf{s}) + \delta\overline{v}_h = u(s_h) + \delta\left[\pi_h u(1) + (1 - \pi_h) u(0) + \delta\overline{v}_h\right]$ from which we get $d_h(\mathbf{s}) = u(s_h)$, $h > m$.

Now, if $u(s_{m+1}) < \dfrac{u(s_m)}{1 - \delta\sum_{i=1}^{\kappa} \pi_i}$ proposer $i < m$ has an incentive to allocate an amount s_{m+1} to legislator $m+1$ instead of choosing m as a coalition partner, since $s_{m+1} < z(\mathbf{s})$ satisfies the demand of legislator $m+1$. For the same reason no pure strategy proposal $\mathbf{z} \in \Delta_{n-2}$ that involves a coalition partner $h > m$ can be equilibrium in these cases since proposers would rather deviate and offer an amount s_m to legislator m. Thus, for certain status quo $\mathbf{s}$, equilibrium requires mixed proposal strategies such that a number of b, $1 \leq b \leq$

m, legislators may become coalition partners of proposers $i < m$ with positive probability.

To construct such mixed strategies, let μ_j^b be the probability with which legislator $j = m, m+1, ..., m+b-1$ is allocated amount $z_b(\mathbf{s})$ by all proposers $i < m$. As a special case, in the pure strategy equilibrium we considered above we have $b = 1$, $u(z_1(\mathbf{s})) = \frac{u(s_m)}{1 - \delta \sum_{i=1}^{\kappa} \pi_i}$, and $\mu_m^1 = 1$. In general, it must be that $u(z_b(\mathbf{s})) + \delta \overline{v}_j = u(s_j) + \delta \left[\pi_j + \left(\sum_{i=1}^{\kappa} \pi_i\right) \mu_j^b u(z_b(\mathbf{s})) + \delta \overline{v}_j\right]$. Solving for μ_j^b we obtain[9]:

$$\mu_j^b = \frac{u(z_b(\mathbf{s})) - u(s_j)}{\delta u(z_b(\mathbf{s})) \sum_{i=1}^{\kappa} \pi_i} \tag{11}$$

Adding the requirement that $\sum_{j=m}^{\kappa+b} \mu_j^b = 1$ on equation (11) we can solve for $u(z_b(\mathbf{s}))$

$$u(z_b(\mathbf{s})) = \frac{\sum_{j=m}^{\kappa+b} u(s_j)}{b - \delta \sum_{i=1}^{\kappa} \pi_i}. \tag{12}$$

Thus, $z_b(\mathbf{s}) = u^{-1}\left(\frac{\sum_{j=m}^{\kappa+b} u(s_j)}{b - \delta \sum_{i=1}^{\kappa} \pi_i}\right)$, and the utility received by any of the b potential coalition partners is an average of the individual stage utilities of these b players inflated by a quantity that depends on the discounted overall probability they are included in the coalition $\delta \sum_{i=1}^{\kappa} \pi_i$.

It remains to ensure that (a) proposer i does not prefer to coalesce with a different legislator, say $h = m + b$, and that (b) probabilities μ_j^b are well defined. Since the demand of player $m + b$ is $d_{m+b}(\mathbf{s}) = u(s_{m+b})$, the first requirement amounts to showing $u(z_b(\mathbf{s})) \leq u(s_{m+b})$. Substituting from (12) for $u(z_b(\mathbf{s}))$, writing $u(s_{m+b})$ as $\sum_{j=m}^{m+b} u(s_j) - \sum_{j=m}^{\kappa+b} u(s_j)$, and re-arranging terms we get $\frac{\sum_{j=m}^{\kappa+b} u(s_j)}{\left(b - \delta \sum_{i=1}^{\kappa} \pi_i\right)} \leq \frac{\sum_{j=m}^{m+b} u(s_j)}{\left(b + 1 - \delta \sum_{i=1}^{\kappa} \pi_i\right)}$, hence

$$u(z_b(\mathbf{s})) \leq u(s_{m+b}) \Longleftrightarrow u(z_b(\mathbf{s})) \leq u(z_{b+1}(\mathbf{s})) \tag{13}$$

Furthermore, to show that μ_j^b are well defined probabilities it suffices to establish that

$$u(z_b(\mathbf{s})) > u(s_j), j = m, m+1, ..., m+b-1 \tag{14}$$

[9] If $\sum_{i=1}^{\kappa} \pi_i = 0$, no $i < m$ ever gets to propose, so this solution is well defined in the relevant cases.

Indeed, equation (11) along with the fact that we have required $\sum_{j=m}^{\kappa+b} \mu_j^b = 1$ ensure $\mu_j^b > 0$. It turns out we can simultaneously ensure the validity of conditions (13) and (14). In particular, condition (13) can equivalently be written as $\sum_{j=m}^{\kappa+b} u(s_j) \leq u(s_{m+b})(b - \delta \sum_{i=1}^{\kappa} \pi_i)$ so that after adding $u(s_{m+b})$ on both sides and re-arranging terms we get an alternative equivalent expression:

(15) $$u(z_b(\mathbf{s})) \leq u(s_{m+b}) \Longleftrightarrow u(z_{b+1}(\mathbf{s})) \leq u(s_{m+b})$$

We then have the following algorithm for the construction of the equilibrium proposal strategies specified in (11) and (12):

1. Start with $b = 1$; if $u(z_1(\mathbf{s})) \leq u(s_{m+1})$ then stop.

2. If $u(z_b(\mathbf{s})) > u(s_{m+b})$, mix with $b+1$ legislators; the contra-positive of (15) ensures that $\mu_j^{b+1} > 0$ for $j = m, ..., m+b$, *i.e.* condition (14) is satisfied.

3. Proceed as above until $u(z_b(\mathbf{s})) \leq u(s_{m+b})$ or until $b+1 > m$. In the latter case, an equilibrium of this type fails to exist.

Thus a necessary condition for an equilibrium to exist, possibly mixing with all m legislators $m, ..., n$, is $u(z_m(\mathbf{s})) \leq 1$. For general $\mathbf{s} \in \Delta_\kappa$ this condition can be written as:

(16) $$\frac{\sum_{i=1}^{n} I_{(0,1)}(s_i) u(s_i)}{m - \delta \sum_{i=1}^{n} I_{\{0\}}(s_i) \pi_i} \leq 1, \mathbf{s} \in \Delta_\kappa$$

It is straightforward to verify that (16) holds for all $\mathbf{s} \in \Delta_\kappa$ if and only if:

(17) $$\frac{m}{m - \delta\widehat{\pi}} u\left(\frac{1}{m}\right) \leq 1$$

where

(18) $$\widehat{\pi} = \max_C \left\{ \sum_{i \in C} \pi_i \mid C \subset N, |C| = \kappa \right\}.$$

A number or remarks are in order:

Remark 1 *Probabilities* μ_j^b *constructed above are unique for any* $\mathbf{s} \in \Delta_\kappa$*, although there is an infinity of mixed strategies by players* $i = 1, 2, ..., \kappa$ *that induce these probabilities.*

Also, given the above probabilities and the continuation values for players derived so far, we can calculate the expected utility function of players for any $\mathbf{x} \in \Delta_{\theta \geq \kappa}$.

Remark 2 *Let* $\mathbf{x} \in \Delta_{\theta \geq \kappa}$ *and without loss of generality let* $i > h \Longrightarrow x_i \geq x_h$*. Then* $U_i(\mathbf{x})$ *can be written as:*

$$U_i(\mathbf{x}) = \begin{cases} u(x_i) + \delta \overline{v}_i & \text{if} \quad i = m+b, ..., n \\ \widetilde{d}(\mathbf{x}) + \delta \overline{v}_i & \text{if} \quad i = m, ...m+b-1 \\ \delta \pi_i u(1 - z_b(\mathbf{x})) + \delta^2 \overline{v}_i & \text{if} \quad i = 1, ..., \kappa \end{cases} \tag{19}$$

where $\widetilde{d}(\mathbf{x}) \equiv u(z_b(\mathbf{x})) = \dfrac{\sum_{j=m}^{m+b-1} u(x_j)}{b - \delta \sum_{i=1}^{\kappa} \pi_i}$ *such that*

$$u(x_{m+b-1}) < \widetilde{d}(\mathbf{x}) \leq u(x_{m+b}) \tag{20}$$

Notice that $\widetilde{d}(\mathbf{x})$, $\mathbf{x} \in \Delta_\kappa$, is the demand of the b legislators with positive probability of receiving funds from a proposer j with allocation $x_j = 0$. When $\mathbf{x} \in \Delta_{\theta > \kappa}$ we have $\widetilde{d}(\mathbf{x}) = 0$. By construction,

Definition 3 $\widetilde{d}(\mathbf{x})$ *is the minimum demand among the majority of* m *players with the highest demands.*

It also follows immediately from the expected utility function in (19) that:

Remark 3 *For* $\mathbf{s} \in \Delta_\kappa$ *we have* $U_i(\mathbf{e}^j) > U_i(\mathbf{s})$, $j \neq i$ *for all* i *with* $s_i = 0$.

In other words, player i with $s_i = 0$ *strictly prefers* a proposal that allocates the whole dollar to another player over a status quo $\mathbf{s} \in \Delta_\kappa$. This is because $\mathbf{s}$ and $\mathbf{e}^j$ imply an identical stream of payoffs for i except for the fact that with $\mathbf{s}$ as the status quo i has to satisfy the demand $\widetilde{d}(\mathbf{s}) > \widetilde{d}(\mathbf{e}^j) = 0$ when proposing, whereas i can extract the whole dollar when the status quo is $\mathbf{e}^j$, $i \neq j$.

In the next few lemmas we establish a number of important properties of the expected utility function in (19).

Lemma 1 *For all $\mathbf{x} \in \Delta_{\theta \geq \kappa}$,(a) $U_i(\mathbf{x})$ is continuous with respect to $\mathbf{x} \in \Delta_{\theta \geq \kappa}$, (b)*

$$x_i \geq x_j \Longrightarrow d_i(\mathbf{x}) \geq d_j(\mathbf{x}), \tag{21}$$

and (c) for every $C \subset N$ with $|C| = \kappa$ and $x_i = 0$ for all $i \in C$, we have $\frac{\partial U_j(\mathbf{x})}{\partial x_j}|_{x_i=0, i \in C} > 0$ for all j such that $u(x_j) > \widetilde{d}(\mathbf{x})$.

Proof. Without loss of generality let $\mathbf{x}$ be such that $i > h \Longrightarrow x_i \geq x_h$. Then utilities $U_i(\mathbf{x})$ are given in equation (19). Note that u is continuous and so part (a) follows if $x_m = 0$ whence $\widetilde{d}(\mathbf{x}) = z_1(\mathbf{x}) = \frac{u(0)}{1 - \delta\pi_m} = 0 \leq u(x_{m+1})$, or for $\mathbf{x}$ such that $\widetilde{d}(\mathbf{x}) < u(x_{m+b})$, since $u(z_b(\mathbf{x}))$ is also continuous. It remains to show continuity for the cases when $\mathbf{x}$ is such that $\widetilde{d}(\mathbf{x}) = u(z_b(\mathbf{x})) = u(x_{m+b}) = ... = u(x_{m+b+g-1}) < u(x_{m+b+g}) \Longleftrightarrow u(z_b(\mathbf{x})) = u(z_{b+1}(\mathbf{x})) = ... = u(z_{b+g}(\mathbf{x})) < u(z_{b+g+1}(\mathbf{x}))$ which, assuming arbitrary g, $1 \leq g \leq m - b$, exhausts all possibilities. In these cases, continuity holds for $U_i(\mathbf{x})$, $i = m + b + g, ..., n$, by the continuity of u. To show continuity for the remaining players, let $\widetilde{d}(\mathbf{x}) = y$. Consider any sequence $\mathbf{x}^k \in \Delta_{\theta \geq \kappa}$ such that $\mathbf{x}^k \longrightarrow \mathbf{x}$. By the continuity of u there exists high enough q so that $u\left(x_i^k\right) > u\left(x_h^k\right)$ if $x_i > x_h$, for all $k > q$. Thus, for $k > q$, it is safe to only consider changes in the ordering of legislators according to their allocation induced by $\mathbf{x}$ for the g

legislators $m+b, ..., m+b+g-1$ with $u(x_{m+b}) = ... = u(x_{m+b+g-1})$. For any subset C of these legislators define $u\left(z_C\left(\mathbf{x}^k\right)\right) = \dfrac{\sum_{i=m}^{m+b-1} u\left(x_i^k\right) + \sum_{i \in C} u\left(x_i^k\right)}{b + |C| - \sum_{i=1}^n I_{\{0\}}(x_i)\pi_i}$. For any k let C_l^k be a subset of $\{m+b, ..., m+b+g-1\}$ with cardinality l such that $i \in C_l^k$, $j \notin C_l^k \Longrightarrow u\left(x_i^k\right) < u\left(x_j^k\right)$ and $u\left(z_{C_l^k}\left(\mathbf{x}^k\right)\right) \leq u\left(x_j^k\right)$ for all $j \notin C_l^k$. Then

$$U_i\left(\mathbf{x}^k\right) = \begin{cases} u\left(z_{C_l^k}\left(\mathbf{x}^k\right)\right) + \delta\overline{v}_i & \text{if} \quad i \in C_l^k \\ u\left(\mathbf{x}_i^k\right) + \delta\overline{v}_i & \text{if} \quad i \notin C_l^k \end{cases} \text{'}i = m+b, ..., m+b+g-1$$

But from $u\left(x_i^k\right) \longrightarrow u(x_i) = y$ for all $i = m+b, ..., m+b+g-1$ we get $u\left(z_C\left(\mathbf{x}^k\right)\right) \longrightarrow \dfrac{\sum_{i=m}^{m+b-1} u(x_i) + |C|\, y}{b + |C| - \sum_{i=1}^n I_{\{0\}}(x_i)\pi_i} = u\left(z_{b+|C|}(\mathbf{x})\right) = y$ for all $C \subset \{m+b, ..., m+b+g-1\}$ and so $U_j\left(\mathbf{x}^k\right) \longrightarrow U_j(\mathbf{x}) = y + \delta\overline{v}_j$ for all $j = m, ..., m+b+g-1$. By the same argument we obtain $U_j\left(\mathbf{x}^k\right) \longrightarrow U_j(\mathbf{x}) = \delta\pi_i u(1 - z_b(\mathbf{x})) + \delta^2\overline{v}_i$ for $j = 1, ..., \kappa$. Part (b) holds by (20), the fact that $d_i(\mathbf{x}) = u(x_i)$ for all i with $u(x_i) > \widetilde{d}(\mathbf{x})$, and since $d_i(\mathbf{x}) = 0$ for all i with $x_i = 0$. Finally, (c) follows directly from $u' > 0$. ■

Following is a technical lemma that provides additional information about the structure of the expected utility functions in (19):

Lemma 2 *Let* $\mathbf{s} \in \Delta_{\theta \geq \kappa}$ *and let* $\widetilde{\pi} = \sum_{i=1}^n I_{\{0\}}(s_i)\pi_i$. *There exist unique* $S_b \in (0,1)$ *for* $b < m$ *and* $S_m = 1$ *such that*

(22) $$\max_{\mathbf{s}} \left\{\widetilde{d}(\mathbf{s}) \mid \widetilde{d}(\mathbf{s}) \leq u(s_{m+b})\right\} = \frac{b}{b - \delta\widetilde{\pi}} u\left(\frac{S_b}{b}\right) = u\left(\frac{1 - S_b}{m - b}\right)$$

(23) $$\frac{b}{b - \delta\widetilde{\pi}} u\left(\frac{S_b}{b}\right) \leq \frac{b+1}{b+1-\delta\widetilde{\pi}} u\left(\frac{S_{b+1}}{b+1}\right), \ 1 \leq b < m,$$

and

(24) $$\max \widetilde{d}(\mathbf{s}) = \frac{m}{m - \delta\widetilde{\pi}} u\left(\frac{1}{m}\right).$$

Proof. W.l.o.g. for each $1 \leq b < m$ we seek a solution to the following program:

$$\max_{\{s_m,\dots,s_n\}} \frac{\sum_{i=m}^{\kappa+b} u(s_i)}{b - \delta\widetilde{\pi}} \text{ subject to} \tag{25}$$

$$\frac{\sum_{i=m}^{\kappa+b} u(s_i)}{b - \delta\widetilde{\pi}} \leq u(s_{m+b}) \tag{26}$$

$$s_j \leq s_{j+1}, j = m, \dots, n-1 \tag{27}$$

$$\sum_{i=m}^{n} s_i = 1 \tag{28}$$

$$s_i \geq 0 \tag{29}$$

Forming the Langrangian we obtain:

$$\begin{aligned} L = & \frac{\sum_{i=m}^{\kappa+b} u(s_i)}{b - \delta\widetilde{\pi}} - \gamma \left(\frac{\sum_{i=m}^{\kappa+b} u(s_i)}{b - \delta\widetilde{\pi}} - u(s_{m+b}) \right) \\ & - \sum_{j=m}^{n-1} \zeta_j (s_j - s_{j+1}) - \lambda \left(\sum_{i=m}^{n} s_i - 1 \right) - \sum_{i=1}^{n} \xi_i (-s_i) \end{aligned}$$

and the first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial s_m} &= (1-\gamma) \frac{1}{b - \delta\widetilde{\pi}} u'(s_m) - \zeta_m - \lambda + \xi_m = 0 \\ \frac{\partial L}{\partial s_j} &= (1-\gamma) \frac{1}{b - \delta\widetilde{\pi}} u'(s_j) - \zeta_j + \zeta_{j-1} - \lambda + \xi_j = 0, j = m+1, \dots, m+b-1 \\ \frac{\partial L}{\partial s_{m+b}} &= \gamma u'(s_{m+b}) - \zeta_{m+b} + \zeta_{m+b-1} - \lambda + \xi_{m+b} = 0 \\ \frac{\partial L}{\partial s_j} &= -\zeta_j + \zeta_{j-1} - \lambda + \xi_j = 0, j = m+b+1, \dots, n-1 \\ \frac{\partial L}{\partial s_n} &= \zeta_{n-1} - \lambda + \xi_n = 0 \\ \gamma \frac{\partial L}{\partial \gamma} &= \lambda \frac{\partial L}{\partial \lambda} = \zeta_j \frac{\partial L}{\partial \zeta_j} = \xi_i \frac{\partial L}{\partial \xi_i} = 0, j = m, \dots, n-1, i = m, \dots, n \end{aligned}$$

$$\gamma, \zeta_j, \xi_i \geq 0, j = m, \dots, n-1, i = m, \dots, n$$

Note that $f_b(S) \equiv \frac{b}{b - \delta\widetilde{\pi}} u\left(\frac{S}{b}\right) - u\left(\frac{1-S}{m-b}\right)$ is a continuous function of S with $f_b' > 0$, $f_b(0) < 0$, and $f_b(1) > 0$ so that there exists a unique $S_b \in (0,1)$ such that $f_b(S_b) = 0$.

Thus, a solution of the form $s_m = ... = s_{m+b-1} = \dfrac{S_b}{b}$ and $s_{m+b} = ... = s_n = \dfrac{1-S_b}{m-b}$ satisfies constraint (26) with equality; it also satisfies constraints (28), (29), and (27), the latter by the fact that $\dfrac{b}{b-\delta\widetilde{\pi}} > 1$ and the monotonicity of u. Denote the above feasible solution by $\mathbf{s}_b$. Then, $\xi_j = 0$, $j = m, ..., n$, $\zeta_{n-1} = \lambda$, $\zeta_j = (n-1-j)\lambda$, $j = m+b+1, ..., n-1$, $\zeta_{m+b-1} = 0$, $\zeta_j = 0$, $j = m, ..., m+b-1$, $\gamma = \dfrac{(n-(m+b)+1)\frac{1}{b-\delta\widetilde{\pi}}u'\left(\frac{S_b}{b}\right)}{(n-(m+b)+1)\frac{1}{b-\delta\widetilde{\pi}}u'\left(\frac{S_b}{b}\right)+u'\left(\frac{1-S_b}{m-b}\right)}$, $\lambda = \dfrac{\frac{1}{b-\delta\widetilde{\pi}}u'\left(\frac{S_b}{b}\right)u'\left(\frac{1-S_b}{m-b}\right)}{(n-(m+b)+1)\frac{1}{b-\delta\widetilde{\pi}}u'\left(\frac{S_b}{b}\right)+u'\left(\frac{1-S_b}{m-b}\right)}$, satisfy the maximization conditions and $\mathbf{s}_b$ is a maximizer of (25). Similarly, $\mathbf{s}_m$ with $s_m = ... = s_n = \dfrac{1}{m}$ is a maximizer in the case $b = m$. To show (23) notice that from $u(z_b(\mathbf{s}_b)) = u\left(\dfrac{1-S_b}{m-b}\right)$ we have $u(z_b(\mathbf{s}_b)) = u(z_{b+1}(\mathbf{s}_b))$ by (13). Since $\mathbf{s}_{b+1}$ maximizes $u(z_{b+1}(\cdot))$ we have $z_{b+1}(\mathbf{s}_b) \leq z_{b+1}(\mathbf{s}_{b+1})$ which proves (23); (24) follows. ■

Notice that we have not characterized proposal strategies – and expected utility functions – over status quo $\mathbf{s} \in \Delta_{\theta<\kappa}$. Thus, we cannot ascertain the optimality of the proposal strategies we characterized above for $\mathbf{s} \in \Delta_{\theta\geq\kappa}$ over all feasible outcomes in $W(\mathbf{s})$. Yet, (19) allows us to check whether these proposals are optima over outcomes in $W(\mathbf{s}) \cap \Delta_{\theta\geq\kappa}$. We do so in the following lemma:

Lemma 3 *Proposal strategies for* $\mathbf{s} \in \Delta_{\theta\geq\kappa}$ *are optimal over alternatives in* $W(\mathbf{s}) \cap \Delta_{\theta\geq\kappa}$ *if*

(30) $$\frac{m}{m-\delta\widehat{\pi}}u\left(\frac{1}{m}\right) \leq u\left(\frac{1}{2}\right), \text{ and}$$

(31) $$\frac{\kappa}{\kappa-\delta\widehat{\pi}}u\left(\frac{z}{\kappa}\right) \leq u(z), \ \ z \leq S_\kappa$$

where S_κ *is defined in Lemma 2 and* $\widehat{\pi}$ *is defined in (18). These conditions are also necessary if* $\pi_i = \dfrac{1}{n}$ *for all* $i \in N$.

Proof. Remark 3 ensures that proposals receive majority approval. Thus, characterized proposals are feasible, and we need show optimality.

Sufficiency: From (19) no proposer can do better than extract the entire dollar, *i.e.* $\mathbf{e}^i \in \arg\max \{U_i(\mathbf{x}) \mid \mathbf{x} \in \Delta_{\theta \geq \kappa}\}, i \in N$, so we only need consider the optimality of proposals for cases when $\mathbf{s} \in \Delta_\kappa$ and proposer i is such that $s_i = 0$. According to equilibrium, in these cases i allocates $z_b(\mathbf{s})$ such that $u(z_b(\mathbf{s})) = \widetilde{d}(\mathbf{s})$ to one of b legislators and retains $1 - z_b(\mathbf{s})$ for herself. Thus, the utility of proposer i is $U_i(0, ..., z_b(\mathbf{s}), 0, ..., 0, 1 - z_b(\mathbf{s}), 0, ...) = u(1 - z_b(\mathbf{s})) + \delta \overline{v}_i$. By part (c) of Lemma 1 these proposals are optima among proposals $\mathbf{z} \in W(\mathbf{s}) \cap \Delta_{\theta \geq m}$. It remains to show that there exists no $\mathbf{x} \in \arg\max \{U_i(\mathbf{z}) \mid \mathbf{z} \in W(\mathbf{s}) \cap \Delta_\kappa\}$ such that $U_i(\mathbf{x}) > U_i(\mathbf{z})$ for all $\mathbf{z} \in W(\mathbf{s}) \cap \Delta_{\theta \geq m}$. To establish a contradiction, assume that such $\mathbf{x}$ exists and w.l.o.g. re-enumerate $\mathbf{x}$ and $\mathbf{s}$ so that $i > j \Longrightarrow x_i \geq x_j$. We shall show a contradiction in a number of steps:

(1) $x_j > 0$ and $U_j(\mathbf{x}) \geq U_j(\mathbf{s})$ for at least some j with $s_j > 0$. If $s_j > 0$ and $x_j = 0$, we have $U_j(\mathbf{x}) < U_j(\mathbf{s})$. Thus if statement does not hold for at least one of m legislators with $s_j > 0$, $\mathbf{x} \notin W(\mathbf{s})$.

(2) $d_{n-1}(\mathbf{x}) > u(x_{n-1})$. Assume the contrary. Then we must have $d_{n-1}(\mathbf{x}) = u(x_{n-1})$ and $d_n(\mathbf{x}) = u(x_n)$ by (21). Consider that among legislators satisfying the criterion in step (1) with minimum $s_j > 0$, say legislator h. Then construct an alternative $\mathbf{w}$ that allocates 0 to all but legislators i and h and, if $h < n$, allocates x_{n-1} to h and $(1 - x_{n-1}) > x_i$ to i or, if $h = n$, allocates x_n to h and $(1 - x_n) > x_i$ to i. Then clearly $U_h(\mathbf{w}) \geq U_h(\mathbf{x})$, while equation (21) ensures $U_i(\mathbf{w}) > U_i(\mathbf{x})$. But $\mathbf{w} \in W(\mathbf{s})$ since $U_l(\mathbf{w}) > U_l(\mathbf{s})$ for all l with $s_l = 0$, which contradicts $U_i(\mathbf{x}) > U_i(\mathbf{z})$ for all $\mathbf{z} \in W(\mathbf{s}) \cap \Delta_{\theta \geq m}$.

(3) $d_{n-1}(\mathbf{x}) < d_n(\mathbf{x})$. Suppose $d_{n-1}(\mathbf{x}) = d_n(\mathbf{x})$ instead. From step (2) we have $\widetilde{d}(\mathbf{x}) = d_m(\mathbf{x}) = ... = d_n(\mathbf{x})$. Lemma 2, and the fact that i optimizes imply $d_i(\mathbf{x}) = d_j(\mathbf{x}) = \frac{m}{m-\delta\widetilde{\pi}}u\left(\frac{1}{m}\right) \leq u\left(\frac{1}{2}\right)$ for any j identified in step (1). But proposal $\mathbf{w}$ with $w_i = w_j = \frac{1}{2}$ is such that $U_j(\mathbf{w}) \geq U_j(\mathbf{x})$, $U_i(\mathbf{w}) > U_i(\mathbf{x})$. We also have $\mathbf{w} \in W(\mathbf{s})$ since $U_h(\mathbf{w}) > U_h(\mathbf{s})$ for all h, $s_h = 0$. The above contradict $U_i(\mathbf{x}) > U_i(\mathbf{z})$ for all $\mathbf{z} \in W(\mathbf{s}) \cap \Delta_{\theta \geq m}$.

From steps (2) and (3), $\mathbf{x}$ is such that $d_h(\mathbf{x}) = z_\kappa(\mathbf{x}) < d_n(\mathbf{x}) = u(x_n)$, $h = m, ..., n-1$. From lemma 2 we have $d_{n-1}(\mathbf{x}) \leq \frac{\kappa}{\kappa-\delta\widehat{\pi}}u\left(\frac{1-x_n}{\kappa}\right)$, and $1-x_n \leq S_\kappa$. From (31) we obtain $\frac{\kappa}{\kappa-\delta\widehat{\pi}}u\left(\frac{1-x_n}{\kappa}\right) \leq u(1-x_n)$. Now proposal $\mathbf{w}$ with $w_i = 1-x_n$, $w_j = x_n$ if $i < n$ or with $w_i = x_n$, $w_j = 1-x_n$ if $i = n$ is such that $U_j(\mathbf{w}) \geq U_j(\mathbf{x})$, $U_i(\mathbf{w}) > U_i(\mathbf{x})$, and $\mathbf{w} \in W(\mathbf{s})$ since $U_h(\mathbf{w}) > U_h(\mathbf{s})$ for all h with $s_h = 0$, which contradicts $U_i(\mathbf{x}) > U_i(\mathbf{z})$ for all $\mathbf{z} \in W(\mathbf{s}) \cap \Delta_{\theta \geq m}$. Thus $\mathbf{x} \in \arg\max\{U_i(\mathbf{z}) \mid \mathbf{z} \in W(\mathbf{s}) \cap \Delta_\kappa\}$ cannot be proposed in an equilibrium proposal strategy.

Necessity when $\pi_i = \frac{1}{n}$ for all $i \in N$: $u(z_m(\mathbf{s})) = \frac{m}{m-\delta\frac{\kappa}{n}}u\left(\frac{1}{m}\right)$, for all $\mathbf{s}$ with m members receiving $\frac{1}{m}$. Suppose $\frac{m}{m-\delta\frac{\kappa}{n}}u\left(\frac{1}{m}\right) > u\left(\frac{1}{2}\right)$, contrary to (30). Then any proposer i with $s_i = 0$ receives $U_i(\mathbf{z}) = u(1-z_m(\mathbf{s})) + \delta\overline{v}_i$ under the equilibrium while she could achieve majority and a higher utility $\frac{m}{m-\delta\frac{\kappa}{n}}u\left(\frac{1}{m}\right) + \delta\overline{v}_i > U_i(\mathbf{z})$ by allocating $\frac{1}{m}$ to herself and κ other legislators. Further assume contrary to (31) that for some $z \leq S_\kappa$, $\frac{\kappa}{\kappa-\delta\frac{\kappa}{n}}u\left(\frac{z}{\kappa}\right) > u(z)$. Continuity of $U_i(\mathbf{s})$ and (22) in lemma 2, ensure there exists $\mathbf{s} \in \Delta_\kappa$ such that $\widetilde{d}(\mathbf{s}) = u(z)$. Then any proposer i with $s_i = 0$ expects $U_i(\mathbf{z}) = u(1-z) + \delta\overline{v}_i$ under the equilibrium while she could achieve majority and a higher utility by allocating $1-z+\varepsilon$ to herself and $\frac{z-\varepsilon}{\kappa}$ to the κ remaining legislators $1, ..., i-1, i+1, ..., m$ with $\varepsilon > 0$ small enough to ensure $\frac{\kappa}{\kappa-\delta\frac{\kappa}{n}}u\left(\frac{z-\varepsilon}{\kappa}\right) = u(z)$. ■

Besides showing (restricted) optimality of proposals, lemma 3 establishes two additional conditions for the existence of equilibrium (equations (30) and (31)) besides (17). Condition (30) ensures that the proposer is better off buying a single legislator among those with positive amount instead of allocating an equal amount $\frac{1}{m}$ to all members in the winning coalition. Similarly, condition (31) ensures that it is not less expensive for the proposer to buy the influence of a single legislator by splitting the same amount equally among κ legislators besides herself.

<<INSERT FIGURE 2 ABOUT HERE>>

Both conditions (30) and (31) are easier to satisfy if the stage utility u displays a smaller degree of risk aversion (or rises at a slower rate), or if the legislature is large, or if the discount factor is small *ceteris paribus*. We leave it to the reader to ascertain that all conditions are satisfied in the case of risk neutrality. Figure 2 depicts a situation when condition (30) is satisfied in the presence of risk aversion. Note that (17) is implied by (30) whereas (30) and (31) are independent. In the form stated in lemma 3 the two conditions are sufficient only and become necessary and sufficient when probabilities of recognition are symmetric. There exists a more general version of these conditions that renders them necessary and sufficient even if recognition probabilities are asymmetric, but we do not pursue this generalization here.

iii. $\theta < \kappa$ To establish an equilibrium then it remains to specify proposal strategies for all $\mathbf{s} \in \Delta_{\theta<\kappa}$, and establish optimality of these and above proposals over all feasible alternatives $\mathbf{z} \in W(\mathbf{s})$. The complexity of the analysis and the multiplicity of cases (see for example Kalandrakis, 2002 for a simpler version of a similar problem) makes it impossible to characterize the equilibrium in closed form. Yet, we are able to provide relatively tight bounds for the range of the model's parameters for which an equilibrium exists. This equilibrium has the property that if the game starts with a status quo $\mathbf{s} \in \Delta_{\theta<\kappa}$ a decision with at least a bare minority of players receiving zero prevails in that period, and the game is subsequently played in the manner we have characterized in closed form in this section. This result and additional properties of the equilibrium are discussed in what follows.

4. RESULTS

In this section we state and discuss the main findings of the analysis. Our chief result is the existence of an equilibrium with the dynamics outlined in the previous section. We provide sufficient conditions for such an equilibrium to exist under risk aversion and/or risk neutrality. Except for the case $n = 5$ when we effectively constrain the discount factor to be smaller than $\frac{5}{8}$, we show that a sufficient condition for existence of equilibrium under risk neutrality is that probabilities of recognition of individual legislators are bounded where these bounds are less restrictive for smaller discount factor or for larger legislatures. If probabilities of recognition are more symmetric than what is required by these bounds, an equilibrium also exists for sufficiently mild concavity of the stage utility u. Specifically:

Proposition 1 *If (a) $u(x) = x$, $n = 5$, $\delta \leq \frac{5}{8}$, $\pi_i \leq \frac{m-\delta-2}{\delta m}$ for all i, or if (b) $u(x) = x$, $n > 5$, $\pi_i \leq \frac{m-\delta-2}{\delta m}$ for all i, or (c) under either of the above cases with $u(x) - x < \varepsilon$ and individual π_i sufficiently smaller than $\frac{m-\delta-2}{\delta m}$, there exists a MPNESUV such that for all measurable $Y \subseteq \Delta$:*

1. *$Y \cap \Delta_{\theta \geq \kappa} = \varnothing \Longrightarrow \mu_i[Y \mid \mathbf{s}] = 0$, for all $\mathbf{s} \in \Delta_{\theta<\kappa}$ and all $i \in N$,*

2. $Y \cap \Delta_{n-2} = \varnothing \Longrightarrow \mu_i [Y \mid \mathbf{s}] = 0$ *for all* $\mathbf{s} \in \Delta_\kappa$ *and all* i *with* $s_i = 0$*,*

3. $Y \cap \{\mathbf{e}^i\} = \varnothing \Longrightarrow \mu_i [Y \mid \mathbf{s}] = 0$ *for all* $\mathbf{s} \in \Delta_\kappa$ *and all* i *with* $s_i > 0$ *or for all* $\mathbf{s} \in \Delta_{\theta>\kappa}$ *and all* $i \in N$*.*

Proof. See Appendix. ■

<<INSERT FIGURE 3 ABOUT HERE>>

The Markov process over policy outcomes induced by this equilibrium is illustrated in Figure 3. We emphasize that the conditions of the theorem are sufficient only, and that the above equilibrium may (and indeed does) exist for a wider range of the parameter space. But even in the case of risk-neutrality we can show via simple examples that an equilibrium does not exist for some parameter values outside these bounds, so that these conditions are not void. We conjecture that in those cases when an equilibrium in the form required by proposition 1 does not exist, there may exist MPNESUV such that outcomes are also absorbed in Δ_{n-1} but with some delay, *i.e.* in more than three periods.

The equilibrium is not unique already from the construction of mixed proposal strategies in subsection 3.ii, as we point out in Remark 1. The fact that all these equilibria are payoff equivalent, poses the question whether the class of MPNESUV for this game are payoff equivalent in analogy to the result of Eraslan, 2002 for the Baron and Ferejohn, 1989, model. Although we do not resolve the question of payoff equivalence[10], the significance of our result lies on a number of properties of the equilibrium, besides its existence.

[10]By payoff equivalence we mean identical expected utility $U_i(\mathbf{s})$ for the same state $\mathbf{s}$. We conjecture that payoff equivalence is less likely to hold, if at all, for less symmetric probabilities of recognition.

First, in equilibrium the support of the steady state distribution of decisions is a finite set. Despite the fact that majority rule induces a social preference cycle that encompasses the whole space of alternatives, only the n outcomes in the set Δ_{n-1} are reached by the committee upon absorption. In addition, after period 3 there is probability $\sum_{i=1}^{n} \pi_i^2$ that the same decision prevails between consecutive periods so that we don't observe perpetual instability of decisions. Cycling and intransitivities of majority rule do not imply inability to decide nor inability to predict outcomes at any given point in time.

Furthermore, convergence to the steady state distribution is fast and takes place within a maximum of three periods. Convergence occurs with certainty in finite time, whereas for $n = 3$ Kalandrakis, 2002 obtains probabilistic convergence in finite time. The difference arises from the fact that with $n = 3$ there are initial status quo and a path of play by nature such that the legislator excluded from last period's allocation is recognized in each period and is unable to extract the whole dollar. Thus, absorption may require more than three periods when $n = 3$.

Besides resolving questions as to the prevalence of chaos in this distributive, dynamic decision making environment, our analysis permits a comparison of equilibrium outcomes with set theoretic solutions for majority rule games. Such comparisons are extensive in the literature. For example, in the one-dimensional space considered by Baron, 1996, a static majority rule core point exists at the median and this also constitutes the long-run absorbing set of the game. Similarly, Banks and Duggan, 2000, show core implementation in the general version of the Baron-Ferejohn bargaining model they consider. In both cases the core is defined on the basis of the stage utilities of legislators as opposed to the discounted sum of the payoffs from the entire sequence of decisions[11]. In the model we consider a core point does not exist, hence a generalization of the notion of the core is in order. One relevant generalization is the uncovered set (Fishburn, 1979, Miller, 1980), which we define below.

Definition 4 *An alternative y covers an alternative x if $y \succ x$ and $x \succ z \Longrightarrow y \succ z$, where $\succ$ represents the strong majority preference relation.*

On the basis of the above covering relation we can define the following sets of alternatives:

Definition 5 *The uncovered set of an alternative $\mathbf{w} \in \Delta$, $UC(\mathbf{w})$, is the set of all elements of Δ that are not covered by $\mathbf{w}$ when the majority preference relation is defined on the basis of the stage utility function u_i.*

Similarly we define:

Definition 6 *The uncovered set of Δ, $UC(\Delta)$, is the set of all elements of Δ that are not covered when the majority preference relation is defined on the basis of the stage utility function u_i.*

Shepsle and Weingast, 1984, show that if voters are sophisticated and for finite amendment agendas starting with a status quo $\mathbf{w}$ the outcome of the vote can only lead to elements of $UC(\mathbf{w})$, *i.e.* alternatives not covered by $\mathbf{w}$. They also argue that competitively constructed agendas can only result in decisions that belong in the uncovered set, $UC(\Delta)$ under amendment agendas.

[11] In the Baron-Ferejohn model the space of possible sequences of decisions is a vector that has the decision at the period when agreement is reached, and an alternative that accrues zero to each legislator as coordinates for all the remaining periods.

The following results show that there is little relation between the above findings and the outcomes that prevail in the dynamic framework we consider.

Proposition 2 *Under the characterized MPNESUV, a decision* $\mathbf{x}$ *may prevail with* $\mathbf{s}$ *as the status-quo and* $\mathbf{x} \notin UC(\mathbf{s})$.

Proof. We can construct an example exploiting Remark 3. Specifically, assume $u(x) = x$, $\pi_i = \frac{1}{n}$, for $i \in N$, and consider $\mathbf{s} = (\varepsilon, ..., \varepsilon, \alpha, \beta - \varepsilon, ..., \beta - \varepsilon)$, with $\alpha = 1 - \kappa\beta$, $\beta > 0$, $\frac{n\alpha}{n - \delta\kappa} < \beta$, and $\varepsilon > 0$ and small. We claim equilibrium proposals[12] are identical to those that prevail for status quo $\mathbf{s}' = (0, ..., 0, \alpha, \beta, ..., \beta) \in \Delta_\kappa$. Specifically, legislators $j = m, ..., n$ successfully propose $\mathbf{e}^j \in \Delta_{n-1}$, while legislators $i = 1, ..., \kappa$ optimize by allocating $z_1(\mathbf{s}') = \frac{\alpha}{1 - \delta\frac{\kappa}{n}} = \frac{n\alpha}{n - \delta\kappa} < \beta - \varepsilon$ to legislator m and retaining the rest of the dollar. Indeed, legislator i's, $i = 1, ..., \kappa$, expected utility from the status quo with the above proposal strategies is given by $U_i(\mathbf{s}) = \varepsilon + \delta\left[\frac{1}{n}\left(1 - \frac{n\alpha}{n - \delta\kappa}\right) + \delta\overline{v}_i\right] < U_i(\mathbf{e}^j) = \delta\overline{v}_i$, $j = m, ..., n$ for sufficiently small ε so that $i = 1, ..., \kappa$ strictly prefer a proposal that allocates the whole dollar to j over the status quo $\mathbf{s}$. But all successful proposals have at most two legislators receiving a positive fraction of the dollar, hence they are covered by $\mathbf{s}$. ■

We obtain this result despite the fact that the agenda within each period is a (degenerate) amendment agenda as in Shepsle and Weingast, 1984. The discrepancy in the findings is of course due to the fact that in our analysis legislative interaction continues after a decision is reached in any one period. Thus, legislators may strictly prefer outcomes that reduce their allocation compared to the status quo.

[12]More precisely, there exists an equilibrium with such proposals, *i.e.* these proposal strategies constitute a fixed point of the mapping used in the proof of proposition 1.

Our second result relies on existing characterization of the uncovered set in the distributive framework:

Proposition 3 *If $u(x) = x$, and irrespective of the initial allocation of the dollar or the discount factor, equilibrium decisions $\mathbf{x}^t \notin UC(\Delta)$ for all $t \geq 3$ under the characterized MPNESUV.*

Proof. Elements in Δ_{n-1} are covered (Epstein, theorem 2, p 88-89, and Maggie Penn, 2001)[13]. ■

Given the fact that we analyze a dynamic game, one may argue that a dynamic definition of the uncovered set is called for, while the above results apply for the stage-defined covering relation. More to the point, the true space of possible alternatives in this game constitutes of lotteries over the stream of decisions $\{\mathbf{x}^t\}_{t=\tau+1}^{+\infty}$, following a decision $\mathbf{x}^\tau$ in arbitrary period τ. Call this space Ψ with generic element $\widetilde{\mathbf{x}}^\tau$. With appropriate assumptions on these lotteries we can define expected utility for player i as $EU_i(\widetilde{\mathbf{x}}^\tau) = u_i(\mathbf{x}^\tau) + \delta E_{\mathbf{x}^\tau}\left[V_i\left(\{\mathbf{x}^t\}_{t=\tau+1}^{+\infty}\right)\right]$. It appears then natural to consider the following definition of the uncovered set:

Definition 7 *The uncovered set of Ψ, $UC(\Psi)$, is the set of all elements of Ψ that are not covered when the majority preference relation is defined on the basis of expected utility $EU_i(\widetilde{\mathbf{x}}^\tau)$.*

Since the equilibrium we characterize exists in the presence of (mild) risk aversion, the stream of payoffs to players may also be *ex ante* Pareto inefficient and belong in the uncovered set according to the above definition. Specifically:

[13] Epstein uses a slightly different definition of the uncovered set, although his theorem applies for the definition of the covering relation we use here. I thank Maggie Penn for pointing this out.

Proposition 4 *Under the characterized MPNESUV outcomes may be ex ante inefficient and equilibrium lotteries may not belong in* $UC(\Psi)$, *for all* $t \geq 3$.

Proof. Assume $\pi_i > 0$ for all $i \in N$ and (mild) risk aversion. Consider allocation $\mathbf{y} \in \Delta$ with $y_i = \pi_i$ for each period $t \geq 3$; it constitutes a Pareto improvement over equilibrium lotteries since $u(\pi_i) > \pi_i > 0$ and we have $\sum_{t=3}^{+\infty} \delta^{t-1} u_i(\mathbf{y}) = \sum_{t=3}^{+\infty} \delta^{t-1} u(\pi_i) > \sum_{t=3}^{+\infty} \delta^{t-1} \pi_i$. Consider then $\widetilde{\mathbf{y}}^\tau \in \Psi$ given by $\widetilde{\mathbf{y}}^\tau \equiv \{\mathbf{y}\}_{t=\tau+1}^{+\infty}$. Since for every decision in period $\tau \geq 3$, $EU_i(\widetilde{\mathbf{y}}^\tau) \geq \sum_{t=3}^{+\infty} \delta^{t-1} \pi_i$ for all i, the equilibrium induced lottery is covered since every alternative in Ψ that is beaten by the equilibrium lottery is also beaten by $\widetilde{\mathbf{y}}^\tau$. ■

Despite the possible inefficiency of equilibrium outcomes and the fact that proposers exercise disproportionate power within each period they are recognized, the overall *ex ante* payoff of players can be very equitable if legislators are patient. In other words, since for many purposes it is the *ex ante* discounted sum of within period payoffs that is relevant for distributional comparisons, the power of the agenda setter may be constrained in the overall game even though it is not constrained within each period.

Unfortunately, this is not always the case. If we calculate the power of an agenda setter in relation to her ideal point defined over the entire stream of payoffs in the game, an analogous result to that of McKelvey, 1976, 1979, obtains for sufficiently small discount factors or for large enough legislature even though voters are strategic. In particular, when $u(x) = x$ and for every initial allocation $\mathbf{x}^0 \in \Delta$ an equilibrium exists if $\pi_i \leq \dfrac{m - \delta - 2}{\delta m}$. Since $m = \kappa + 1$ we have the following corollary of proposition 1:

Corollary 1 (Dictatorship) *For* $\kappa \geq \dfrac{1 + 2\delta}{1 - \delta}$, $\pi_i = 1$, $u(x) = x$ *and any initial allocation* $\mathbf{x}^0 \in \Delta$ *there exists a MPNESUV such that* i *extracts the whole dollar in every period* $t \geq 3$.

Finally, our analysis produces different comparative statics than those induced in models without recurrent decisions. One instance of this discrepancy is the role of risk-aversion that is completely different in this model than in the model of Baron and Ferejohn, 1989. In particular, Harrington, 1990, shows that higher degrees of risk aversion increase the power of the proposer, while we have established that the characterized equilibrium collapses when risk aversion is high.

We now establish a similar counter-result with regard to the effect of probabilities of recognition on the demands of individual legislators. Eraslan, 2002, shows that in the Baron and Ferejohn model legislators with higher probabilities of recognition are more expensive coalition partners *ceteris paribus*. This is because if a proposal is rejected in the current period, then these legislators will be able to extract the surplus in the following period with higher probability. Probabilities of recognition have the opposite effect in the model we analyze. Legislators with high probability of being recognized are more willing to accept a bad proposal in the current period, since it allows them to extract more of the dollar in the following period. We shall show this for the case of risk-neutrality which is the case comparable to the analysis of Eraslan. Specifically:

Proposition 5 *If $u(x) = x$, the characterized MPNESUV is such that for $i \neq j$ with $s_i = s_j$ and $\pi_i > \pi_j$, $d_i(\mathbf{s}) \leq d_j(\mathbf{s})$ for every $\mathbf{s} \in \Delta$. If $d_i(\mathbf{s}) < d_j(\mathbf{s})$ then $\mathbf{s} \in \Delta_{\theta<\kappa}$.*

Proof. For $\mathbf{s} \in \Delta_{\theta \geq \kappa}$ we immediately have from (19) that $s_i = s_j \Longrightarrow d_i(\mathbf{s}) = d_j(\mathbf{s})$. For $\mathbf{s} \in \Delta_{\theta<\kappa}$, assume $d_i(\mathbf{s}) > d_j(\mathbf{s})$ to show a contradiction. To reduce the notational burden below we write d_h for $d_h(\mathbf{s})$, $h \in N$. We will also use the notation:

- D_l, the amount allocated by proposer l to the remaining players,

- μ_{hl}, the probability that player $h \neq l$ allocates a positive amount to l, and

- $m_h^l(\mathbf{z})$, the conditional probability distribution for the proposal strategy of h, conditional on $z_l = 0$.

We shall derive a contradiction using a series of steps:

(1) If l proposes $\mathbf{z}$, $U_q(\mathbf{s}) = U_q(\mathbf{z})$ for all $q \neq l$ with $z_q > 0$. W.l.o.g. assume $h > g \Longrightarrow z_h \geq z_g$. By step (7) in the proof of Proposition 1 proposer l optimizes by minimizing D_l. Thus part (c) of lemma 1 guarantees the validity of the claim for $q = m + b, ..., n, q \neq l$ with $u(z_q) \geq \widetilde{d}(\mathbf{z}) = z_b(\mathbf{z})$. Now let S be the sum of the amounts allocated to players $q = m, .., m + b - 1$. We have $z_b(\mathbf{z}) = \dfrac{S}{b - \delta \sum_{j=m}^{\kappa+b} \pi_j}$. Now, assume contrary to the claim that $\widetilde{d}(\mathbf{z}) > d_h(\mathbf{s})$ for some player $h = m, .., m + b - 1$ and let $d_h(\mathbf{s}) = \min\{d_j(\mathbf{s}) \mid j = m, ..., m + b - 1\} < z_b(\mathbf{z})$. Then l may allocate $\widehat{x}$ to player h such that $d_h(\mathbf{s}) = \dfrac{\widehat{x}}{1 - \delta \sum_{j=1}^{\kappa} \pi_j}$, $\widetilde{x} = z_b(\mathbf{z}) > \widehat{x}$ to the remaining $b - 1$ players, and maintain amounts allocated to the other players. Clearly this alternative proposal obtains majority approval with a smaller cost. Indeed $(b-1)\widetilde{x} + \widehat{x} = (b-1)\dfrac{S}{b - \delta \sum_{j=1}^{\kappa} \pi_j} + d_h(\mathbf{s})\left(1 - \delta \sum_{j=1}^{\kappa} \pi_j\right) < S \Longleftrightarrow d_h(\mathbf{s})\left(1 - \delta \sum_{j=1}^{\kappa} \pi_j\right) < S - (b-1)\dfrac{S}{b - \delta \sum_{j=1}^{\kappa} \pi_j} \Longleftrightarrow d_h(\mathbf{s}) < \dfrac{S}{b - \delta \sum_{j=1}^{\kappa} \pi_j}$, which is true so that we have a contradiction due to the fact we assumed $d_h(\mathbf{s}) < u(z_b(\mathbf{z}))$.

(2) The expected utility of player $l, l \in N$ can be written as

$$U_l(\mathbf{s}) = s_l + \delta \left[\pi_l (1 - D_l) + \sum_{h \neq l} \pi_h \left(\mu_{hl} d_l - (1 - \mu_{hl}) \int m_h^l(d\mathbf{z})\, \delta \pi_l \widetilde{d}(\mathbf{z}) \right) + \delta \overline{v}_l \right]. \tag{32}$$

By steps (6) and (7) in the proof of proposition 1, l's utility when proposing is equal to $(1 - D_l) + \delta \overline{v}_l$. If $h \neq l$ proposes $\mathbf{z}$, then (19) and step (1) guarantee that l receives utility $d_l + \delta \overline{v}_l$ if $z_l > 0$. Finally, if $z_l = 0$, $l's$ utility is $-\delta \pi_l \widetilde{d}(\mathbf{z}) + \delta \overline{v}_l$ (with $\widetilde{d}(\mathbf{z})$ possibly zero).

(3) If $d_l > 0, l \in N$ then

$$d_l = \frac{s_l - \delta\pi_l D_l - \sum_{h\neq l}\pi_h(1-\mu_{hl})\int m_h^l(d\mathbf{z})\,\delta\pi_l\widetilde{d}(\mathbf{z})}{1-\delta\sum_{h\neq l}\pi_h\mu_{hl}} \tag{33}$$

$$d_l = \frac{s_l - \delta\pi_l(D_l + d_l) - \sum_{h\neq l}\pi_h(1-\mu_{hl})\int m_h^l(d\mathbf{z})\,\delta^2\pi_l\widetilde{d}(\mathbf{z})}{1-\delta\left(\pi_l + \sum_{h\neq l}\pi_h\mu_{hl}\right)} \tag{34}$$

Both versions are derived by substituting from equation (32) in $U_l(\mathbf{s}) = d_l + \delta\overline{v}_l$. For (34) add and subtract $\delta\pi_l d_l$ in the resultant equation and partially solve for d_l leaving term $-\delta\pi_l d_l$ in the right hand side.

(4) If $\mu_{ij} = 1$, $D_j + d_j \leq D_i + d_i$. Let $\mathbf{z}$ be any proposal by i, with $\widetilde{\pi} = \sum_{h=1}^n I_{\{0\}}(z_h)\pi_h$. We distinguish three cases. Case 1: $z_j = d_j$; then j can propose $\mathbf{w}$ with $w_h = z_h, h \neq i, j$, $w_i = d_i$, and $w_j = d_j + (1 - D_i) - d_i$. Then $D_j = 1 - (d_j + (1 - D_i) - d_i) \iff D_j + d_j = D_i + d_i$. Case 2: $z_j = d_j(1 - \delta\widetilde{\pi})$. Let $l \in \arg\min\{d_h \mid h \neq j,\ z_h > 0\}$. By (1) and (19) we have $d_l > d_j$. Then j can propose $\mathbf{w}$ with $w_h = z_h, h \neq i, j, l$, $w_i = d_i$ if $i \neq l$, $w_l = d_l(1 - \delta\widetilde{\pi})$, and retain the rest of the dollar. Then, if D is the sum of allocations that are unchanged between $\mathbf{z}$ and $\mathbf{w}$, when $i \neq l$ we have $D_i = D + d_l + d_j(1 - \delta\widetilde{\pi})$ and $D_j \leq D + d_l(1 - \delta\widetilde{\pi}) + d_i$. Hence, $D_j + d_j \leq D_i + d_i \implies d_l \geq d_j$ which is true. Similarly, if $i = l$ we have $D_i = D + d_j(1 - \delta\widetilde{\pi})$ and $D_j \leq D + d_i(1 - \delta\widetilde{\pi})$, whence $D_j + d_j \leq D_i + d_i \implies d_i \geq d_j$ which is true by the working hypothesis. Case 3: $d_h = d_j = \dfrac{\sum_{h\in C} z_h}{b - \delta\widetilde{\pi}}$ for a coalition of $b \geq 2$ legislators (including j) in C with $z_h > 0, h \in C$. Then, j can successfully propose $\mathbf{w}$ with $w_h = z_h, h \notin C \cup \{i\}$, $w_i = d_i$, $w_h = d_j\dfrac{(b - 1 - \delta\widetilde{\pi})}{b - 1}$, $h \in C - \{j\}$, and retain the rest of the dollar. We may then write $D_i = D + d_j(b - \delta\widetilde{\pi})$ and $D_j \leq D + d_j(b - 1 - \delta\widetilde{\pi}) + d_i$, whence $D_j + d_j \leq D_i + d_i$ $\implies d_i \leq d_i$ which is also true.

(5) If $\mu_{ij} < 1$ then $D_j \leq D_i - \delta(\pi_i - \pi_j)\widetilde{d}(\mathbf{z})$ for all $\mathbf{z}$ with $z_j = 0$ proposed by i, and $\mu_{ji} = 0$. Since $\mu_{ij} < 1$ there exists some $\mathbf{z}$ with $z_j = 0$, $\sum_{l=1}^n I_{\{0\}}(z_l)\pi_l = \widetilde{\pi}$ proposed by i. Let

there be a set C with $b \leq \kappa$ players such that $d_h = \widetilde{d}(\mathbf{z}), h \in C$ and $\sum_{h\in C} z_h = \widetilde{d}(\mathbf{z})(b - \delta\widetilde{\pi})$. Then construct $\mathbf{w}$ such that $w_h = z_h, h \neq i, j$, and $z_h > \widetilde{d}(\mathbf{z})$, $w_h = \widetilde{d}(\mathbf{z})(b - \delta(\widetilde{\pi} - \pi_j + \pi_i))$ for $h \in C$, $w_i = 0$, and the rest of the dollar to j. $\mathbf{w}$ maintains majority approval since the demand of all players previously receiving positive funds besides i is satisfied. Thus $D_j \leq D_i - \widetilde{d}(\mathbf{z})(b - \delta\widetilde{\pi}) + \widetilde{d}(\mathbf{z})(b - \delta(\widetilde{\pi} - \pi_j + \pi_i))$ which simplifies to the first part we wish to establish. It remains to show $\mu_{ji} = 0$. Suppose j proposes $\mathbf{x}$ with $x_i > 0$ to get a contradiction. Let $D_j = D + x_i$. Since $d_j < d_i$ by the working hypothesis, i may propose $\mathbf{y}$ with $y_h = x_h$ if $h \neq i, j$ and $y_j < x_i$ and still obtain majority approval, whence $D_j > D_i$ contradicting our earlier finding.

(6) $d_j > 0$. Suppose not. Then $d_j = 0$ and by step (6) in the proof of Proposition 1 equation (32) reduces to $U_j(\mathbf{s}) = s_j + \delta[\pi_j(1 - D_j) + \delta\overline{v}_j] \leq \delta\overline{v}_j \iff s_j \leq \delta\pi_j D_j$. From $d_i > d_j$ we have $d_i > 0$. Then, since $1 - \delta\sum_{h\neq i}\pi_h\mu_{hi} \geq \delta\pi_i$, we obtain from equation (33) that $d_i \leq \dfrac{s_i - \delta\pi_i D_i}{\delta\pi_i} \iff \delta\pi_i(D_i + d_i) \leq s_i$. From steps (4) and (5) we have $D_j \leq (D_i + d_i)$. Thus, from $s_j = s_i$ and $\pi_i > \pi_j$ we get $\delta\pi_i(D_i + d_i) \leq s_i \Longrightarrow \delta\pi_j D_j < s_j$, a contradiction.

(7) $\mu_{hj} > \mu_{hi}$ and $z_i > 0 \Longrightarrow z_j > 0$ for all $\mathbf{z}$ proposed by $h \neq i, j$. Obvious if h's proposal $\mathbf{z} \in \Delta_{\theta>\kappa}$ (by step (6) of proposition 1 either all or none of h's optimal proposals belong in $\Delta_{\theta>\kappa}$) since $d_i > d_j$ and h minimizes D_h. To show a contradiction in the remaining cases, suppose h proposes $\mathbf{z} \in \Delta_\kappa$ with $z_i > 0, z_j = 0$. Construct $\mathbf{w}$ such that $w_j = z_i, w_i = 0$ with all remaining allocations identical to those in $\mathbf{z}$. We have $\pi_i > \pi_j \Longrightarrow \sum_{h=1}^n I_{\{0\}}(z_h)\pi_h > \sum_{h=1}^n I_{\{0\}}(w_h)\pi_h$ hence j strictly prefers $\mathbf{w}$ over $\mathbf{z}$ since $d_j < d_i$. Similarly, all remaining players besides i previously approving $\mathbf{z}$ approve $\mathbf{w}$. By the strict preference of j and step (1) h can successfully propose an allocation with a smaller cost than that allocated under $\mathbf{z}$,which contradicts optimality of $\mathbf{z}$.

(8) If i proposes $\mathbf{z}$ with $z_j = 0$ and j proposes $\mathbf{w}$ with $w_i = 0$, $\widetilde{d}(\mathbf{w}) \geq \widetilde{d}(\mathbf{z})$. Let $\sum_{l=1}^{n} I_{\{0\}}(w_l)\pi_l = \widetilde{\pi}$. We have $D_j = \sum_{h \neq j, w_h > d(\mathbf{w})} w_i + (b - \delta\widetilde{\pi})\widetilde{d}(\mathbf{w})$ for some b players. Then, by a similar argument to that in step (5) $D_i \leq \sum_{h \neq j, w_h > d(\mathbf{w})} w_i + (b - \delta(\widetilde{\pi} - \pi_i + \pi_j))\widetilde{d}(\mathbf{w})$ $\iff D_i - D_j \leq \delta(\pi_i - \pi_j)\widetilde{d}(\mathbf{w})$. But by step (5) we have $\delta(\pi_i - \pi_j)\widetilde{d}(\mathbf{z}) \leq D_i - D_j$ and the claim follows.

We are now ready to establish a contradiction emanating from the working hypothesis $d_i > d_j$. We distinguish two cases. Case 1: $\mu_{ij} = 1$. Step (6) guarantees we can represent both d_i, d_j by equation (34). We then have $1 - \delta\left(\pi_i + \sum_{h \neq i} \pi_h \mu_{hi}\right) > 1 - \delta\left(\pi_j + \sum_{h \neq j} \pi_h \mu_{hj}\right)$ by step (7), and the fact that $\mu_{ij} = 1$. Also, from $\pi_j < \pi_i$, step (7), and the fact that $\mu_{ij} = 1$, we obtain $-\sum_{h \neq i} \pi_h (1 - \mu_{hi}) \int m_h^i(d\mathbf{z}) \delta^2 \pi_i \widetilde{d}(\mathbf{z}) < -\sum_{h \neq j} \pi_h (1 - \mu_{hj}) \int m_h^j(d\mathbf{z}) \delta^2 \pi_j \widetilde{d}(\mathbf{z})$. Thus, using the above and equation (34) we have $d_i > d_j \implies s_i - \delta\pi_i(D_i + d_i) > s_j - \delta\pi_j(D_j + d_j) \iff \pi_i(D_i + d_i) < \pi_j(D_j + d_j)$ which is false by step (4) and the fact that $\pi_j < \pi_i$. Case 2: $\mu_{ij} < 1$. From step (7) and the fact that $\mu_{ji} = 0$ we have $1 - \delta\sum_{h \neq i} \pi_h \mu_{hi} > 1 - \delta\sum_{h \neq j} \pi_h \mu_{hj}$. Also, from $\pi_j < \pi_i$, step (7), and the fact that $\mu_{ji} = 0$, we obtain $-\sum_{h \neq i,j} \pi_h (1 - \mu_{hi}) \int m_h^i(d\mathbf{z}) \delta^2 \pi_i \widetilde{d}(\mathbf{z}) < -\sum_{h \neq j,i} \pi_h (1 - \mu_{hj}) \int m_h^j(d\mathbf{z}) \delta^2 \pi_j \widetilde{d}(\mathbf{z})$. Finally, we have $-\pi_j(1 - \mu_{ji}) \int m_j^i(d\mathbf{z}) \delta^2 \pi_i \widetilde{d}(\mathbf{z})$ $\leq -\pi_i(1 - \mu_{ij}) \int m_i^j(d\mathbf{z}) \delta^2 \pi_j \widetilde{d}(\mathbf{z}) \iff \int m_j^i(d\mathbf{z}) \widetilde{d}(\mathbf{z}) \geq (1 - \mu_{ij}) \int m_i^j(d\mathbf{z}) \widetilde{d}(\mathbf{z})$ which is true from step (8). From the above and using equation (33) we get that $d_i > d_j \implies s_i - \delta\pi_i D_i > s_j - \delta\pi_j D_j \iff \pi_i D_i < \pi_j D_j$ which is false by step (5). ■

5. CONCLUSIONS

We analyzed a dynamic majority rule bargaining game over a distributive policy space with an endogenous reversion point. Although subgame perfect equilibrium may fail to exist for games in the class we analyze, we established existence of a (refined) Markov Perfect Nash equilibrium. The equilibrium produces a number of novel and in many instances counter-intuitive findings.

These results do not depend on the way we resolve indifference at the critical voting period when alternatives that allocate zero to more than a majority of legislators prevail for the first time. As we point out in Remark 3 or in the proof of proposition 2, legislators that vote *yes* in these cases and receive zero *strictly* prefer the proposal over the status quo despite the fact that their allocation is reduced because they can extract more of the dollar in subsequent periods.

The reader may object that less counter-intuitive legislative behavior can prevail by simply removing the restriction to Markovian strategies. We point out, though, that much of the significance of our findings emanates from the fact that they differ from results in other studies that impose the same or similar equilibrium restrictions. This discrepancy exists both when we compare these findings with the intuition emerging from related analyses that assume a different policy space but the same institutions (e.g. Baron, 1996, Ferejohn, McKelvey, Packel, 1984, Baron and Herron, 1999), as well as when we consider the same policy domain but different institutional arrangements (Baron and Ferejohn, 1989, Eraslan, 2002).

The first comparison suggests that the nature of legislative interaction is not independent of the underlying policy space. The second comparison casts doubt on our ability to draw valid conclusions from models of legislative politics that assume interaction ceases after a decision is reached.

In sum, we answer some and open even more questions on the dynamics of legislative interaction. One open question is the general existence of equilibrium in such dynamic bargaining games. The results in Kalandrakis, 2002, suggest that general existence may not obtain with a continuous space of decisions without an expansion of the state space. It is also possible that additional MPNESUV with different payoff implications may exist in the game we analyze. We deem it more likely that a stable outcome may be supported in such an equilibrium for sufficiently high degrees of risk aversion, or if probabilities of recognition are sufficiently non-symmetric. Finally, it is worth exploring the effect of more competitive agenda formation processes within each period, such as a version of the open rule considered by Baron and Ferejohn, 1989.

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APPENDIX

In this Appendix we prove the main result (Proposition 1).

Proof of Proposition 1. First note that if $u(x) - x$ is small then conditions (30) and (31) are satisfied. Hence, the proposal strategies analyzed in sections (3.i-3.ii) are consistent with proposition 1 and specify the expected utility functions in (19) for all $\mathbf{s} \in \Delta_{\theta \geq \kappa}$. Now consider $\mathbf{s} \in \Delta_{\theta < \kappa}$; part 1 requires that proposals only belong in $\Delta_{\theta \geq \kappa}$. Informally, then, our proof strategy proceeds as follows: consistent with the above requirement we force players to propose alternatives in $\Delta_{\theta \geq \kappa}$ when $\mathbf{s} \in \Delta_{\theta < \kappa}$. Since we have derived utilities for outcomes in $\Delta_{\theta \geq \kappa}$, then for any conjecture of such play we can measure the expected utility of legislators if they reject these proposals and keep the status quo for only one more period, whence we can analyze optimal voting strategies. Thus we proceed to establish (A) that this restricted game that is afterwards played as described in section 3, has a fixed point in proposal and voting strategies. Then we establish that (B) forcing players to propose only in $\Delta_{\theta \geq \kappa}$ has no consequence, i.e. the fixed point in (A) is such that proposers indeed optimize by only considering proposals in $\Delta_{\theta \geq \kappa}$.

We show (A) by applying Glicksberg's (1952) theorem for every $\mathbf{s} \in \Delta_{\theta<\kappa}$. Verifying the conditions of the theorem in our case differs in a number of ways from related literature (for example Banks and Duggan, 2000, 2001). First, for particular conjectures about proposal strategies in Φ, some proposers may not wish to propose anything in $\Delta_{\theta \geq \kappa}$ over the status quo. Still, we require that they optimize among alternatives in $\Delta_{\theta \geq \kappa}$ and at least a bare minority of κ other players prefer their proposal. The fact that we ignore the vote of the proposer has no effect at (or near) the fixed point as we show in (B). Yet, as we show in Lemma 5, showing that the proposer can find κ other coalition partners is not straightforward due to the nature of the utilities in (19). Hence some restrictions on the applicability of the equilibrium emerge from this requirement. Also, although the feasible set from which the proposer optimizes is upper-hemicontinuous, it fails lower-hemicontinuity and so the typical strategy of establishing upper-hemicontinuity of the best response correspondence by invoking the theorem of the Maximum does not work. Instead, we prove upper-hemicontinuity of proposal strategies directly. This accounts for the additional restrictions on the parameter range in addition to those emanating from Lemma 5.

More formally, consider any $\widehat{\mu} \in \Phi \equiv [\wp(\Delta_{\theta \geq \kappa})]^n$ with coordinate $\widehat{\mu}_i[\mathbf{s}]$ corresponding to the proposal strategy of individual i; if these strategies involve proposals that are accepted, then the expected utility of player i in the event the proposal is rejected is given by $\widehat{U}_i(\widehat{\mu}, \mathbf{s}) \equiv u(s_i) + \delta \sum_{j=1}^{n} \pi_j \int U_i(\mathbf{z}) \widehat{\mu}_j [d\mathbf{z} \mid \mathbf{s}]$, where $U_i(\mathbf{z})$ are as in (19). Define the acceptance set of player i as $\widehat{A}_i(\widehat{\mu}, \mathbf{s}) = \left\{\mathbf{x} \in \Delta_{\theta \geq \kappa} \mid U_i(\mathbf{x}) \geq \widehat{U}_i(\widehat{\mu}, \mathbf{s})\right\}$. Also define the *coalitional winset of* $\mathbf{s}$ *for player* i *given* $\widehat{\mu}$ as the finite union of finite intersections of individual acceptance sets of players other than i over all coalitions $\Gamma_i \equiv \{C \subset N - i \mid |C| = \kappa\}$, *i.e.* $\widehat{W}_i(\widehat{\mu}, \mathbf{s}) \equiv \cup_{C\in\Gamma_i}\left[\cap_{j\in C}\widehat{A}_j(\widehat{\mu}, \mathbf{s})\right]$.

Define the correspondence $M_i(\widehat{\mu}, \mathbf{s}) \equiv \arg\max\left\{U_i(\mathbf{x}) \mid \mathbf{x} \in \widehat{W}_i(\widehat{\mu}, \mathbf{s})\right\}, i \in N$, and let $M(\widehat{\mu}, \mathbf{s}) \equiv M_1(\widehat{\mu}, \mathbf{s}) \times ... \times M_n(\widehat{\mu}, \mathbf{s})$. Let $B_i(\widehat{\mu}, \mathbf{s}) = \wp(M_i(\widehat{\mu}, \mathbf{s}))$. As discussed above, we wish to show that (A) the correspondence $B : \Phi \twoheadrightarrow \Phi$ defined by $B(\widehat{\mu}, \mathbf{s}) \equiv B_1(\widehat{\mu}, \mathbf{s}) \times ... \times B_n(\widehat{\mu}, \mathbf{s})$ has a fixed point $\widehat{\mu}^* = B(\widehat{\mu}^*, \mathbf{s})$, and that (B) $\widehat{\mu}^*$ is such that for all $\mathbf{y} \in \Delta_{\theta<\kappa}$ and for all $C \subset N$, $|C| = m$ there exists $\mathbf{x} \in \Delta_{\theta\geq\kappa}$ such that $\mathbf{x} \in \cap_{j\in C}\widehat{A}_j(\widehat{\mu}^*, \mathbf{y})$. Then, proposal strategies $\widehat{\mu}^*$ and those specified in sections (3.i-3.ii), determine utility functions and acceptance sets that constitute MPNESUV. This is because (B) implies that optimal proposals can only exist in $\Delta_{\theta\geq\kappa}$, so that $\widehat{\mu}^*$ and strategies specified in sections (3.i-3.ii) are optimal over the entire winset $W(\mathbf{s})$. We show (A) in a series of steps:

(1) $\widehat{W}_i(\widehat{\mu}, \mathbf{s})$ is non-empty for all $i \in N$. We establish this using two lemmas:

Lemma 4 *For the utilities in (19) and any coalition C, with $|C| > m$, and $\sum_{i\in C}\pi_i = 1-\pi_e$*

$$\max\left\{\sum_{i\in C}[U_i(\mathbf{x}) - \overline{v}_i] \mid \mathbf{x} \in \Delta_{\theta\geq\kappa}, \pi_{i\in C}\right\} =$$

$$= (1-\pi_e)[\delta u(1-w) - (1+\delta)] + m\frac{m}{m-\delta}u\left(\frac{1}{m}\right) \tag{35}$$

with $w = u^{-1}\left(\frac{m}{m-\delta}u\left(\frac{1}{m}\right)\right)$, *and for* $|C| = m$

$$\max\left\{\sum_{i\in C}[U_i(\mathbf{x}) - \overline{v}_i] \mid \mathbf{x} \in \Delta_{\theta\geq\kappa}, \pi_{i\in C}\right\} =$$

$$= m\frac{m}{m-\delta\pi_e}u\left(\frac{1}{m}\right) - (1-\pi_e) \tag{36}$$

Proof. From standard application of the Kuhn-Tucker conditions and for any δ, π_i,

$i \in C$ we obtain

$$\max \left\{ \sum_{i \in C} \left[U_i \left(\mathbf{x} \right) - \overline{v}_i \right] \mid \mathbf{x} \in \Delta_{\theta > \kappa} \right\} = \kappa u \left(\frac{1}{\kappa} \right) - \left(1 - \pi_e \right)$$

and the maximum for $\mathbf{x} \in \Delta_{\theta > \kappa}$ is attained for $\mathbf{x} \in \Delta_{\kappa+1}$. Now consider the maximum of $\sum_{i \in C} \left[U_i \left(\mathbf{x} \right) - \overline{v}_i \right]$ for $\mathbf{x} \in \Delta_{\kappa}$, holding δ, π_i, constant. Assume w.l.o.g. that $C = \{n, ..., l\}$, $1 \leq l \leq \kappa$. Assuming that the maximum is attained when positive amounts are allocated among members in C only, let $i > j \implies x_i \geq x_j$, $\pi_0 = \sum_{i=l}^{\kappa} \pi_i$, and $\widetilde{\pi} = \sum_{i=1}^{\kappa} \pi_i$ so that the objective function can be written as $\sum_{i \in C} \left[U_i \left(\mathbf{x} \right) - \overline{v}_i \right] = \sum_{i=l}^{n} \left[U_i \left(\mathbf{x} \right) - \overline{v}_i \right] = \delta \pi_0 u \left(1 - z_b \left(\mathbf{x} \right) \right) + b u \left(z_b \left(\mathbf{x} \right) \right) + \sum_{i=m+b}^{n} u \left(x_i \right) - \left(1 - \pi_e \right) - \delta \pi_0$ when $u \left(x_{m+b-1} \right) < u \left(z_b \left(\mathbf{x} \right) \right) \leq u \left(x_{m+b} \right)$. While $\sum_{i=1}^{n} \left[U_i \left(\mathbf{x} \right) - \overline{v}_i \right]$ is continuous by lemma 1 it is not differentiable at points such that $u \left(z_b \left(\mathbf{s} \right) \right) = u \left(s_{m+b} \right)$; yet, for each $b = 1, ..., m$ we can maximize:

$$\max_{\{x_m, ..., x_n\}} \left[\delta \pi_0 u \left(1 - z_b \left(\mathbf{x} \right) \right) + b u \left(z_b \left(\mathbf{x} \right) \right) + \sum_{i=m+b}^{n} u \left(x_i \right) - \left(1 - \pi_e \right) - \delta \pi_0 \right] \text{ s.t.}$$

$$u \left(z_b \left(\mathbf{x} \right) \right) = \frac{\sum_{j=m}^{m+b-1} u \left(x_j \right)}{b - \delta \widetilde{\pi}}$$

$$u \left(z_b \left(\mathbf{x} \right) \right) \leq u \left(x_{m+b} \right)$$

$$\sum_{i=m}^{n} x_i = 1$$

$$x_{j+1} \geq x_j, j = m, ..., n-1$$

$$x_j \geq 0, j = m, ..., n$$

It is straightforward to show that the maximum for the above program is attained only if $x_i = \dfrac{1-S}{m-b}, i = m+b, ..., n-1$ and $x_j = \dfrac{S}{b}$ $j = m, ..., m+b-1$, and $S \leq S_b$ where S_b is defined in lemma 2. Else, we can increase $\sum_{i=m+b}^{n} u \left(x_i \right)$ by appropriate re-allocation of amounts x_j, x_i while keeping $u \left(z_b \left(\mathbf{x} \right) \right)$ (and $\delta \pi_0 u \left(1 - z_b \left(\mathbf{x} \right) \right)$) constant. Thus, from lemma

2 we have $\dfrac{bu\left(\dfrac{S}{b}\right)}{b-\delta\widetilde{\pi}} \leq u\left(\dfrac{1-S}{m-b}\right)$, $\dfrac{S}{b} \leq \dfrac{1-S}{m-b}$, $S \leq S_b$ and we can equivalently write the above program (omitting the constant terms) as:

$$\max_{\{S\}} \left[\delta\pi_0 u\left(1-u^{-1}\left(\frac{bu\left(\frac{S}{b}\right)}{b-\delta\widetilde{\pi}}\right)\right)+b\frac{bu\left(\frac{S}{b}\right)}{b-\delta\widetilde{\pi}}+(m-b)\,u\left(\frac{1-S}{m-b}\right)\right] \text{ s.t.}$$

$$S \geq 0$$

$$S \leq S_b$$

Formulating the Langrangian we obtain:

$$L = \delta\pi_0 u\left(1-u^{-1}\left(\frac{bu\left(\frac{S}{b}\right)}{b-\delta\widetilde{\pi}}\right)\right)+b\frac{bu\left(\frac{S}{b}\right)}{b-\delta\widetilde{\pi}}+$$

$$(m-b)\,u\left(\frac{1-S}{m-b}\right)-\gamma\left(S-S_b\right)-\lambda\left(-S\right)$$

and using the fact that $\dfrac{\partial u^{-1}(y)}{\partial y} = \dfrac{1}{u'\left(u^{-1}(y)\right)}$ the first order conditions are:

$$\frac{\partial L}{\partial S} = -\delta\pi_0 u'\left(1-u^{-1}\left(\frac{bu\left(\frac{S}{b}\right)}{b-\delta\widetilde{\pi}}\right)\right)u'\left(u^{-1}\left(\frac{bu\left(\frac{S}{b}\right)}{b-\delta\widetilde{\pi}}\right)\right)^{-1}\frac{u'\left(\frac{S}{b}\right)}{b-\delta\widetilde{\pi}}$$

$$+\frac{b}{b-\delta\widetilde{\pi}}u'\left(\frac{S}{b}\right)-u'\left(\frac{1-S}{m-b}\right)-\gamma+\lambda=0$$

$$\gamma\frac{\partial L}{\partial \gamma} = \lambda\frac{\partial L}{\partial \lambda} = 0$$

$$\gamma \geq 0$$

Setting $S = S_b$, $\lambda = 0$, we get $\gamma = \dfrac{1}{(b-\delta\widetilde{\pi})}\left(b-\delta\pi_0\dfrac{u'\left(1-\dfrac{1-S_b}{m-b}\right)}{u'\left(\dfrac{1-S_b}{m-b}\right)}\right)u'\left(\dfrac{S_b}{b}\right)-u'\left(\dfrac{1-S_b}{m-b}\right)$ where use is made of the fact that $\dfrac{b}{b-\delta\widetilde{\pi}}u\left(\dfrac{S_b}{b}\right) = u\left(\dfrac{1-S_b}{m-b}\right)$. It remains to verify that

$\gamma \geq 0$; since $u'' < 0$ and $\frac{S_b}{b} < \frac{1-S_b}{m-b}$ it suffices to show $\frac{1}{(b-\delta\widetilde{\pi})}\left(b-\delta\pi_0\frac{u'\left(1-\frac{1-S_b}{m-b}\right)}{u'\left(\frac{1-S_b}{m-b}\right)}\right) \geq 1 \iff \pi_0 u'\left(1-\frac{1-S_b}{m-b}\right) \leq \widetilde{\pi} u'\left(\frac{1-S_b}{m-b}\right)$ which is true since $u'' < 0$, $\frac{1-S_b}{m-b} \leq \frac{1}{2}$ by (23) and (31), and $\pi_0 \leq \widetilde{\pi}$. Call the allocation that corresponds to the above solution $\mathbf{x}_b$. By (13) $\mathbf{x}_q$ is a feasible allocation for the corresponding maximization when $b = q+1$, so that $\sum_{i=l}^{n}\left[U_i\left(\mathbf{x}_q\right)-\overline{v}_i\right] < \sum_{i=l}^{n}\left[U_i\left(\mathbf{x}_{q+1}\right)-\overline{v}_i\right]$. Specifically $\sum_{i=l}^{n} U_i\left(\mathbf{x}_1\right) > \kappa u\left(\frac{1}{\kappa}\right)-(1-\pi_e)$, and $\max\left\{\sum_{i=l}^{n}\left[U_i\left(\mathbf{x}\right)-\overline{v}_i\right] \mid \mathbf{x} \in \Delta, x_1 = ... = x_\kappa = 0\right\} = \sum_{i=l}^{n}\left[U_i\left(\mathbf{x}_m\right)-\overline{v}_i\right]$ where

$$\sum_{i=l}^{n}\left[U_i\left(\mathbf{x}_m\right)-\overline{v}_i\right] = \left[\delta\pi_0 u\left(1-z_m\left(\mathbf{x}_m\right)\right)+mu\left(z_m\left(\mathbf{x}_m\right)\right)-(1-\pi_e)-\delta\pi_0\right] \tag{37}$$

If $l = m$, then $\pi_0 = 0$ and (36) follows. Now consider $l < m$; we have

$$\frac{\partial\sum_{i=l}^{n}\left[U_i\left(\mathbf{x}_m\right)-\overline{v}_i\right]}{\partial\pi_0} = \delta u\left(1-z_m\left(\mathbf{x}_m\right)\right)+$$
$$\delta\frac{m-\delta\pi_0\frac{u'\left(1-z_m\left(\mathbf{x}_m\right)\right)}{u'\left(z_m\left(\mathbf{x}_m\right)\right)}}{m-\delta\left(\pi_0+\pi_e\right)}u\left(z_m\left(\mathbf{x}_m\right)\right)-\delta > 0$$

so (37) is increasing in π_0, hence the maximum is attained for $\pi_0 = 1-\pi_e$ for which (37) reduces to (35).

We have derived the above assuming no $i \notin C$ receive a positive amount, and we shall now show this is a restriction of no consequence. For $|C| > m$ suppose there exists $\mathbf{y} \in \Delta_{\theta\geq\kappa}$ with $y_i > 0$, $i \notin C$ such that $\sum_{i\in C}\left[U_i\left(\mathbf{y}\right)-\overline{v}_i\right] > (1-\pi_e)\left[\delta u\left(1-w\right)-(1+\delta)\right]+m\frac{m}{m-\delta}u\left(\frac{1}{m}\right)$. From equation (24) as it applies to w we infer $\sum_{i\notin C}\left[U_i\left(\mathbf{y}\right)-\overline{v}_i\right] \geq \delta\pi_e u\left(1-w\right)-(1+\delta)\pi_e$. Then, $\mathbf{y}$ is such that $\sum_{i\in N}\left[U_i\left(\mathbf{y}\right)-\overline{v}_i\right] > \left[\delta u\left(1-w\right)-(1+\delta)\right]+m\frac{m}{m-\delta}u\left(\frac{1}{m}\right)$ which is impossible since $\max_{\mathbf{x}}\sum_{i\in N}\left[U_i\left(\mathbf{x}\right)-\overline{v}_i\right] = \left[\delta u\left(1-w\right)-(1+\delta)\right]+m\frac{m}{m-\delta}u\left(\frac{1}{m}\right)$. For $|C| = m$, let one legislator not in C receive a positive amount.

From (19) the legislator in C with zero allocation contributes a non-positive amount in the sum, and similar arguments as above show that the maximum amount attainable is $\kappa\frac{m}{m-\delta}u\left(\frac{1}{m}\right)-(1-\pi_e)<m\frac{m}{m-\delta\pi_e}u\left(\frac{1}{m}\right)-(1-\pi_e)\Longleftrightarrow\frac{\kappa}{(m-\delta)}<\frac{m}{(m-\delta\pi_e)}$ which is true for all $\delta<1$. Obviously sum is even smaller if two or more legislators outside C receive positive funds. ■

The second lemma is:

Lemma 5 $\widehat{W}_i(\widehat{\mu},\mathbf{s})$ *is non-empty for all* $\widehat{\mu}\in\Phi$ *if (a)* $u(x)=x$ *and* $\pi_i\leq\frac{m-\delta}{\delta m}$ *for all* $i\in N$, *or (b) if* $u(x)$ *is sufficiently close to risk neutrality and individual probabilities of recognition are sufficiently smaller than* $\frac{m-\delta}{\delta m}$.

Proof. Define the demand of legislator i as $\widehat{d}_i\equiv\max\left\{\widehat{U}_i(\widehat{\mu},\mathbf{s})-\delta\overline{v}_i,0\right\}$. Although for any given pair $(\widehat{\mu},\mathbf{s})$ it is possible that $\widehat{d}_i>1$ for some i, it suffices to show that there exists a coalition C, with $|C|=m$ and $\sum_{j\in C}u^{-1}\left(\widehat{d}_j\right)\leq 1$. If this is the case, then $\sum_{j\in C\backslash\{i\}}u^{-1}\left(\widehat{d}_j\right)\leq 1$, for all $j\in C$. The latter condition ensures that we can allocate $z_j=u^{-1}\left(\widehat{d}_j\right)$ to κ of the legislators in C other than the proposer i (possibly in C) and zero to the rest. The corresponding allocation $\mathbf{z}$ is such that $U_j(\mathbf{z})\geq\widehat{U}_j(\widehat{\mu},\mathbf{s})$ for all these κ legislators in $C\backslash\{i\}$ as can be ascertained from (19), hence $\mathbf{z}\in\widehat{W}_i(\widehat{\mu},\mathbf{s})$. To show that $\sum_{j\in C}u^{-1}\left(\widehat{d}_j\right)\leq 1$ is true for some such coalition C, without loss of generality re-label legislators so that $i>j\Longrightarrow\widehat{d}_i>\widehat{d}_j$. Let $l=\arg\min\left\{i\mid\widehat{d}_i>0\right\}$. Obviously proof follows trivially if $l>m$. In the remaining cases we need establish $\sum_{i=l}^{m}u^{-1}\left(\widehat{d}_i\right)\leq 1$. We will do so by contradiction. As in Lemma 4 let $\pi_e=\sum_{i=1}^{l-1}\pi_i$. Suppose $\sum_{i=l}^{m}u^{-1}\left(\widehat{d}_i\right)=A\geq 1$. Then we have

(1) $\sum_{i=l}^{n}\left[\widehat{U}_i(\widehat{\mu},\mathbf{s})-\delta\overline{v}_i\right]\leq(n-l+1)\,u\left(\frac{1}{n-l+1}\right)+\delta\max\left\{\sum_{i=l}^{n}\left[U_i(\mathbf{x})-\overline{v}_i\right]\mid\mathbf{x}\in\Delta_{\theta\geq\kappa},\pi_{i=l,\ldots,n}\right\}$.

We have $\widehat{U}_i(\widehat{\mu},\mathbf{s})-\delta\overline{v}_i=u(s_i)+\delta\left[\sum_{j=1}^{n}\pi_j\int U_i(\mathbf{y})\,d\widehat{\mu}_j[\mathbf{y}\mid\mathbf{s}]-\overline{v}_i\right]$ and additive separability of $\widehat{U}_i(\widehat{\mu},\mathbf{s})$ implies that for any given $\widehat{\mu}$, $\sup\left\{\left[\widehat{U}_i(\widehat{\mu},\mathbf{s})-\delta\overline{v}_i\right] \mid \widehat{\mu}\in\left[\wp\left(\Delta_{\theta\geq\kappa}\right)\right]^n, \mathbf{s}\in\Delta_{\theta<\kappa}\right\}$ is attained for $\mathbf{s}$ such that exactly $(n-l+1)$ legislators receive $\frac{1}{n-l+1}$ each, which accounts for the first part of the sum, $(n-l+1)\,u\left(\frac{1}{n-l+1}\right)$. The remaining follows since it must be that $\sum_{i=l}^n\left[\sum_{j=1}^n \pi_j \int U_i(\mathbf{y})\,d\widehat{\mu}_j[\mathbf{y}\mid\mathbf{s}]-\overline{v}_i\right]\leq \max\left\{\sum_{i=l}^n\left[U_i(\mathbf{x})-\overline{v}_i\right]\mid \mathbf{x}\in\Delta_{\theta\geq\kappa},\pi_i, i=l,...,n\right\}$.

(2) $\min\left\{\sum_{i=l}^n \widehat{d}_i \mid \sum_{i=l}^m u^{-1}\left(\widehat{d}_i\right)=A, A\geq 1\right\}=(n-l+1)\,u\left(\frac{1}{m-l+1}\right)$[14]. We have

$$\min_{\left\{A,d_l,...,d_n\right\}}\sum_{i=l}^n \widehat{d}_i$$

$$\widehat{d}_j\leq\widehat{d}_{j+1}, j=l,...,n-1$$

$$\sum_{i=l}^m u^{-1}\left(\widehat{d}_i\right)=A$$

$$A\geq 1$$

$$\widehat{d}_m\leq 1$$

$$\widehat{d}_l\geq 0, j=l,...,m$$

Formulating the Langrangian we get

$$L=\sum_{i=l}^n \widehat{d}_i-\sum_{j=l}^{n-1}\zeta_j\left(\widehat{d}_{j+1}-\widehat{d}_j\right)-\lambda\left(\sum_{i=l}^m u^{-1}\left(\widehat{d}_i\right)-A\right)$$
$$-\beta(A-1)-\eta\left(\widehat{d}_l\right)-\xi\left(1-\widehat{d}_m\right)$$

[14]The inverses are well defined since $\widehat{d}_m<1$ as we prove later.

using the fact that $\dfrac{\partial u^{-1}(y)}{\partial y} = \dfrac{1}{u'(u^{-1}(y))}$ the minimization conditions are

$$\frac{\partial L}{\partial \widehat{d}_l} = 1 + \zeta_l - \lambda \frac{1}{u'\left(u^{-1}\left(\widehat{d}_l\right)\right)} - \eta = 0$$

$$\frac{\partial L}{\partial \widehat{d}_j} = 1 - \zeta_{j-1} + \zeta_j - \lambda \frac{1}{u'\left(u^{-1}\left(\widehat{d}_j\right)\right)} = 0, j = l+1, ..., m-1$$

$$\frac{\partial L}{\partial \widehat{d}_m} = 1 - \zeta_{m-1} + \zeta_m - \lambda \frac{1}{u'\left(u^{-1}\left(\widehat{d}_m\right)\right)} + \xi = 0$$

$$\frac{\partial L}{\partial \widehat{d}_j} = 1 - \zeta_{j-1} + \zeta_j = 0, j = m+1, ..., n$$

$$\frac{\partial L}{\partial \widehat{d}_n} = 1 - \zeta_{n-1} = 0$$

$$\frac{\partial L}{\partial A} = \lambda - \beta = 0$$

$$\zeta_j \frac{\partial L}{\partial \zeta_j} = \beta \frac{\partial L}{\partial \beta} = \eta \frac{\partial L}{\partial \eta} = \xi \frac{\partial L}{\partial \xi} = 0$$

$$\zeta_j, \beta, \eta, \xi \geq 0$$

For solution $A = 1$, $\widehat{d}_j = u\left(\frac{1}{m-l+1}\right)$, $j = l, ..., n$, we have $\zeta_j = (n-j) > 0$, $j = m, ..., n-1, \xi = 0, \eta = 0$, $\zeta_m - j = (n-m)\left(1 - \dfrac{h}{(m-l+1)}\right) > 0, h = 1, ..., m-l$, $\lambda = \beta = \lambda = \dfrac{(n-l+1)}{(m-l+1)} u'\left(\frac{1}{m-l+1}\right) > 0$ so that the minimum is $(n-l+1)\, u\left(\frac{1}{m-l+1}\right)$.

(3) $(n-l+1)\, u\left(\frac{1}{m-l+1}\right) \leq (n-l)\, u\left(\frac{1}{m-l}\right)$. For any $l < m$, an allocation with $\widehat{d}_l = 0, \widehat{d}_{l+i} = u\left(\frac{1}{m-l}\right)$ for $i = 1, ..., n$ is also feasible but not optimal.

(4) $(n-l+1)\, u\left(\frac{1}{n-l+1}\right) \geq (n-l)\, u\left(\frac{1}{n-l}\right)$. Same argument as in step (4) for the max-

imization program

$$\max_{\{x_l,\ldots,x_n\}} \sum_{i=l}^{n} u\left(x_i\right)$$
$$\sum_{i=l}^{n} x_i = 1$$
$$x_i \geq 0$$

We are now ready to show the contradiction emanating from the working hypothesis $\sum_{i=l}^{m} u^{-1}\left(\widehat{d}_i\right) = A \geq 1$. For every l we must have the sum of players' demands larger than the minimum possible, *i.e.* $\sum_{i=l}^{n}\left[\widehat{U}_i\left(\widehat{\mu},\mathbf{s}\right) - \delta\overline{v}_i\right] \geq (n-l+1)\,u\left(\frac{1}{m+1-l}\right)$. First, consider $l=m$, in which case after substituting and using steps (1), (2), and (36) we obtain

$$\begin{gathered}\sum_{i=m}^{n}\left[\widehat{U}_i\left(\widehat{\mu},\mathbf{s}\right) - \delta\overline{v}_i\right] > m \Longleftrightarrow \\ mu\left(\frac{1}{m}\right) + \delta\left[m\frac{m}{m-\delta\pi_e}u\left(\frac{1}{m}\right) - (1-\pi_e)\right] > m \Longleftrightarrow \\ u\left(\frac{1}{m}\right) > \frac{m+\delta\left(1-\pi_e\right)}{m} \Longrightarrow u\left(\frac{1}{m}\right) \geq \frac{\kappa}{n}\end{gathered}$$

which is false for sufficiently mild degrees of risk aversion. Similarly, for $l<m$ due to steps (1), (2), and equation (35) of Lemma 4 we have

$$\begin{aligned}&(n-l+1)\,u\left(\frac{1}{n-l+1}\right) + \delta\left[(1-\pi_e)\left[\delta u\left(1-w\right) - (1+\delta)\right] + m\frac{m}{m-\delta}u\left(\frac{1}{m}\right)\right] \\ &> (n-l+1)\,u\left(\frac{1}{m+1-l}\right)\end{aligned}$$

Using steps (3) and (4) and the fact that $\pi_e = 0$ if $l=1$ we obtain two relevant versions: for $l=2$

(38)
$$(n-1)\,u\left(\frac{1}{n-1}\right) + \delta\left[(1-\pi_e)\left[\delta u\left(1-w\right) - (1+\delta)\right] + m\frac{m}{m-\delta}u\left(\frac{1}{m}\right)\right] > (n-1)\,u\left(\frac{1}{\kappa}\right)$$

and for $l = 1$

$$nu\left(\frac{1}{n}\right) + \left[\delta u\left(1-w\right) - \left(1+\delta\right)\right] + m\frac{m}{m-\delta}u\left(\frac{1}{m}\right) > nu\left(\frac{1}{m}\right) \tag{39}$$

In the case of risk neutrality we get $\pi_e = \pi_1 > \frac{m-\delta}{\delta m}$ from (38) and $1 > \frac{n}{m}$ from (39). The latter implies a contradiction for $l = 1$, while $\pi_i \leq \frac{m-\delta}{\delta m}$ ensures a contradiction for $l = 2, ..., \kappa$. Notice though that if each π_i is sufficiently smaller than $\frac{m-\delta}{\delta m}$ then both inequalities are strict, hence result continues to hold if players are mildly risk averse such that $u\left(x\right) - x < \varepsilon$, for some $\varepsilon > 0$. ■

With the above we established step (1).

(2) $\widehat{W}_i\left(\widehat{\mu}, \mathbf{s}\right)$ is compact. This follows from the continuity of $U_i\left(\mathbf{x} \in \Delta_{\theta \geq \kappa}\right)$ (Lemma 1) and the compactness of $\Delta_{\theta \geq \kappa}$.

(3) $\widehat{A}_i\left(\widehat{\mu}, \mathbf{s}\right)$ has closed graph for all $i \in N$. We have $Gr\widehat{A}_i = \left\{\left(\widehat{\mu}, \mathbf{x}\right) \in \Phi \times \Delta_{\theta \geq \kappa} \mid \mathbf{x} \in \widehat{A}_i\left(\widehat{\mu}, \mathbf{s}\right)\right\}$. If $Gr\widehat{A}_i = \emptyset$ then it is closed. Otherwise, suppose by way of contradiction that there exists a sequence $\left(\widehat{\mu}^k, \mathbf{x}^k\right) \in Gr\widehat{A}_i$ such that $\left(\widehat{\mu}^k, \mathbf{x}^k\right) \longrightarrow \left(\widehat{\mu}, \mathbf{x}\right) \notin Gr\widehat{A}_i$. Since $\left[U_i\left(\mathbf{x}\right) - \widehat{U}_i\left(\widehat{\mu}, \mathbf{s}\right)\right]$ is jointly continuous, $\left[U_i\left(\mathbf{x}^k\right) - \widehat{U}_i\left(\widehat{\mu}^k, \mathbf{s}\right)\right] \longrightarrow \left[U_i\left(\mathbf{x}\right) - \widehat{U}_i\left(\widehat{\mu}, \mathbf{s}\right)\right]$. But then $\mathbf{x} \in \widehat{A}_i\left(\widehat{\mu}, \mathbf{s}\right)$, since otherwise $\left[U_i\left(\mathbf{x}^k\right) - \widehat{U}_i\left(\widehat{\mu}^k, \mathbf{s}\right)\right] \geq 0$ for all k and $\left[U_i\left(\mathbf{x}\right) - \widehat{U}_i\left(\widehat{\mu}, \mathbf{s}\right)\right] < 0$.

(4) $\widehat{W}_i\left(\widehat{\mu}, \mathbf{s}\right)$ has closed graph. Finite unions and intersections of closed sets are closed.

(5) $\widehat{W}_i\left(\widehat{\mu}, \mathbf{s}\right)$ is upper-hemicontinuous. The Closed Graph Theorem (Aliprantis and Border, 16.12, p. 529) applies.

The following two steps ((6) and (7)) are used in order to establish that proposers' maximization is equivalent to maximizing the amount of funds they receive. This is then used to establish upper-hemicontinuity of proposers' best responses in (8).

(6) If $\left[\widehat{U}_j\left(\widehat{\mu}, \mathbf{s}\right) - \delta\overline{v}_j\right] \leq 0$ for all $j \in C, |C| \leq \kappa$, then for all $i \neq j \in C$, $\mathbf{y} \in \arg\max\left\{U_i\left(\mathbf{z}\right) \mid \mathbf{z} \in \widehat{W}_i\left(\widehat{\mu}, \mathbf{s}\right)\right\} \Longrightarrow \mathbf{y} \in \Delta_{\kappa+|C|}$. First consider $|C| = 1$. Suppose claim

does not hold; then i allocates a positive amount to κ other legislators and herself (since she optimizes). Without loss of generality, denote this allocation $\mathbf{x} \in \Delta_\kappa$ with $0 < x_m \leq x_{m+1} \leq ... \leq x_n$, and let $u(x_{m+b-1}) < u(z_b(\mathbf{x})) \leq u(x_{m+b})$. It suffices to show that $\sum_{i=m}^{\kappa+b} x_i > (b-1) z_b(\mathbf{x})$, since from (19) we have that allocating $z_b(\mathbf{x})$ to $b-1$ (excluding say h) of b legislators $m, ..., m+b-1$, by non-decreasing remaining amounts $x_{m+b}, ..., x_n$, and allocating the surplus for herself, i obtains the support of all legislators that previously supported her except for h, as well as the support of j so that she maintains the support of κ others (from the fact that i optimizes we deduce that j cannot be one of legislators $m+b, ..., n$ hence at most $j = h$ and above hold; if $b = 1$ and $i = m$ any other legislator can be excluded with the same effect). In the case of risk neutrality we have $\sum_{i=m}^{\kappa+b} x_i > (b-1) z_b(\mathbf{x}) \implies \sum_{i=m}^{\kappa+b} x_i > (b-1) \dfrac{\sum_{i=m}^{\kappa+b} x_i}{b - \delta\widetilde{\pi}} \iff \delta\widetilde{\pi} < 1$ which is true for every $\pi \in \Delta$, $\delta < 1$. Since the inequality is strict, same holds if $u(x) - x < \varepsilon$ for sufficiently small ε. But *a fortiori* a proposal $\mathbf{x} \in \Delta_m$ is not optimal when $|C| > 1$. Now, for proposals in $\Delta_{\theta > m}$ maximization of proposer's utility is equivalent to minimization of the amount allocated to the remaining legislators, so only legislators with positive demand receive a positive amount.

(7) If $\left[\widehat{U}_j(\widehat{\mu}, \mathbf{s}) - \delta\overline{v}_j\right] \geq 0$ for all $j \in N - \{i\}$, then $\mathbf{y} \in \arg\max\left\{U_i(\mathbf{z}) \mid \mathbf{z} \in \widehat{W}_i(\widehat{\mu}, \mathbf{s})\right\}$ $\implies U_i(\mathbf{y}) > \dfrac{m}{m - \delta(1-\pi_i)} u\left(\dfrac{1}{m}\right) + \delta\overline{v}_i$. Consider the case of risk neutrality first. It suffices to show that there exist κ other legislators, say in coalition $C \in \Gamma_i$, such that $\sum_{j \in C} \max\left\{\widehat{U}_j(\widehat{\mu}, \mathbf{s}) - \delta\overline{v}_j\right\} \leq 1 - \dfrac{1}{m - \delta(1-\pi_i)}$. If so, then player i can allocate an amount equal to $\left\{\widehat{U}_j(\widehat{\mu}, \mathbf{s}) - \delta\overline{v}_j\right\}$ to each player in C and retain an amount $\dfrac{1}{m - \delta(1-\pi_i)}$ for herself. This allocation obviously belongs in $\widehat{W}_i(\widehat{\mu}, \mathbf{s})$ and ensures utility $\dfrac{1}{m - \delta(1-\pi_i)} + \delta\overline{v}_i$ for player i by (24) of lemma 2. From equation (35) of lemma 4 and since $2\left(1 - \dfrac{1}{m - \delta(1-\pi_i)}\right) > 2\left(1 - \dfrac{1}{m-\delta}\right)$ it suffices to show that $1 + \delta\left[(1-\pi_i)\left[\delta\left(1 - \dfrac{1}{m-\delta}\right) - (1+\delta)\right] + \dfrac{m}{m-\delta}\right] \leq$

$2\left(1-\frac{1}{m-\delta}\right)$ so that result holds for $\pi_i \leq \frac{m-\delta-2}{\delta m}$, for all i. Since it cannot be that $\pi_i < \frac{1}{n}$ for all i, we must have $\frac{m-\delta-2}{\delta m} \geq \frac{1}{n}$ which is equivalent to $\delta \leq \frac{5}{8}$ if $m = 3$ and is strictly true for $m > 3$. For mild degrees of risk aversion the same holds since the inequality above is strict.

(8) $M_i(\widehat{\mu}, \mathbf{s})$ is upper-hemicontinuous. Non-emptiness and upper-hemicontinuity of $\widehat{W}_i(\widehat{\mu}, \mathbf{s})$ ensures that the maximum in $M_i(\widehat{\mu}, \mathbf{s}) = \arg\max\left\{U_i(\mathbf{z}) \mid \mathbf{z} \in \widehat{W}_i(\widehat{\mu}, \mathbf{s})\right\}$ is well defined and it suffices to show that $M_i(\widehat{\mu}, \mathbf{s})$ has closed graph. Suppose not; then there exists a sequence $\left(\widehat{\mu}^k, \mathbf{x}^k\right) \in GrM_i = \{(\widehat{\mu}, \mathbf{x}) \in \Phi \times \Delta_{\theta \geq \kappa} \mid \mathbf{x} \in M_i(\widehat{\mu}, \mathbf{s})\}$ such that $\left(\widehat{\mu}^k, \mathbf{x}^k\right) \longrightarrow (\widehat{\mu}, \mathbf{x}) \notin GrM_i$. Upper-hemicontinuity of $\widehat{W}_i(\widehat{\mu}, \mathbf{s})$ guarantees $(\widehat{\mu}, \mathbf{x}) \in GrW_i$. Then from the fact that $(\widehat{\mu}, \mathbf{x}) \notin GrM_i$ we deduce there exists $\mathbf{y} \in \arg\max\left\{U_i(\mathbf{z}) \mid \mathbf{z} \in \widehat{W}_i(\widehat{\mu}, \mathbf{s})\right\}$ such that $U_i(\mathbf{y}) - U_i(\mathbf{x}) = \eta > 0$. Optimality of $\mathbf{y}$ ensures $\mathbf{y} \in \arg\max\left\{U_i(\mathbf{z}) \mid U_j(\mathbf{z}) \geq \widehat{U}_j(\widehat{\mu}, \mathbf{s}), j \in C'\right\}$ for some $C' \in \Gamma_i$. Since i optimizes by allocating a positive amount to herself (proposals $\mathbf{g}$ with $g_i = 0$ are such that $U_i(\mathbf{g}) \leq \delta \overline{v}_i$ whereas by Lemma 5 she can obtain strictly higher utility), at most κ other legislators receive positive funds and $1 - \sum_{j \in C'} y_j = y_i > 0$. In fact, steps (6) and (7), as well as part (c) of Lemma 1 ensure that legislator i optimizes by maximizing the amount of funds she allocates to herself, *i.e.* by minimizing $\sum_{j \in C'} y_j$. By the continuity of $\widehat{U}, U$ there exists large enough $k > \overline{k}$ such that $\left|\widehat{U}_j\left(\widehat{\mu}^k, \mathbf{s}\right) - \widehat{U}_j(\widehat{\mu}, \mathbf{s})\right| = \eta_j^k < \min\left\{\frac{\eta}{2\kappa}, \frac{y_i}{\kappa}\right\}$ for all $j \in C'$ and $\left|U_i\left(\mathbf{x}^k\right) - U_i(\mathbf{x})\right| < \frac{\eta}{2}$. Then for each $k > \overline{k}$ it is feasible to construct $\mathbf{y}^k \in \widehat{W}_i\left(\widehat{\mu}^k, \mathbf{s}\right)$ with $y_j^k = y_j + \eta_j^k \longrightarrow y_j$ for all $j \in C'$ and $y_i^k = y_i - \sum_{j \in C'}^k \eta_j^k$ so that $\left|U_i\left(\mathbf{y}^k\right) - U_i(\mathbf{y})\right| < \frac{\eta}{2}$ and $\left|U_i\left(\mathbf{x}^k\right) - U_i(\mathbf{x})\right| < \frac{\eta}{2}$. Then $U_i\left(\mathbf{y}^k\right) > U_i\left(\mathbf{x}^k\right)$ which contradicts $\left(\widehat{\mu}^k, \mathbf{x}^k\right) \in GrM_i$. The above hold for mild risk aversion as well.

(9) B has a fixed point $\widehat{\mu}^* = B(\widehat{\mu}^*, \mathbf{s})$. B is non-empty, upper-hemcontinuous, compact, and convex valued and so has a fixed point by Glicksberg (1952).

This completes the proof of (A) and it remains to show (B). For any $\mathbf{y} \in \Delta_{\theta<\kappa}$, let $U_i^*(\mathbf{y}) \equiv \widehat{U}_i(\widehat{\mu}^*, \mathbf{y})$, where $\widehat{\mu}^*$ is – a – fixed point for $\mathbf{s} = \mathbf{y}$. We first claim:

(10) $U_j^*(\mathbf{y}) - \delta\overline{v}_j < 1$ for all $j \in N$. Suppose not. Then no $i \neq j$ optimizes by allocating a positive amount to j. Thus j's utility $U_j^*(\mathbf{y})$ must be smaller than $u(y_j) + \delta[\pi_j u(1) + \delta\overline{v}_j] = u(y_j) + \delta\overline{v}_j$, and we have a contradiction since $u(y_j) < 1$.

For the next step define the demand in the usual way, *i.e.* $d_j^*(\mathbf{y}) = \max\left\{U_j^*(\mathbf{y}) - \delta\overline{v}_j, 0\right\}$.

(11) If $U_i^*(\mathbf{y}) - \delta\overline{v}_i < 0$, for $k \geq 1$ legislators, then $U_i^*(\mathbf{y}) \geq u(y_i) + \delta[\pi_i u(1 - D) + \delta\overline{v}_i]$, where $D = \min\left\{\sum_{j \in C} u^{-1}\left(d_j^*(\mathbf{y})\right) \mid C \in \Gamma_i\right\}$, with the inequality possibly strict only if $k = 1$. By step (6) all legislators other than i propose alternatives in $\Delta_{\theta>\kappa}$ and never allocate a positive amount to i. Thus, i receives a positive amount only when she proposes. If $k > 1$, again by steps (6), i maximizes by minimizing the amount allocated to $\kappa - 1$ other players, whereas the same applies by (7) if $k = 1$. In the latter case, i can obtain higher utility than $u(1 - D)$ when she proposes as the demand d of another legislator can be satisfied by allocating x with $d > u(x) > 0$ such that $d = \dfrac{u(x)}{1 - \delta\left(\sum_{i \in C'} \pi_i\right)}$, where C' is the set of members that receive zero.

We can now prove (B) *i.e.* that $\widehat{\mu}^*$ is such that for all $\mathbf{y} \in \Delta_{\theta<\kappa}$ and for all $C \subset N$, $|C| = m$ there exists $\mathbf{x} \in \Delta_{\theta\geq\kappa}$ such that $\mathbf{x} \in \cap_{j \in C}\widehat{A}_j(\widehat{\mu}^*, \mathbf{y})$. Without loss of generality assume $i > j \implies U_i^*(\mathbf{y}) - \delta\overline{v}_i \geq U_j^*(\mathbf{y}) - \delta\overline{v}_j$. Consider the case of risk neutrality first. Let $l = \min\{i \in N \mid U_i^*(\mathbf{y}) - \delta\overline{v}_i > 0\}$. Assuming $l \leq m$[15] it suffices to show that for every coalition $C \subset \{l, ..., n\}$ with $|C| = m$, $h = \min\{i \in C\}$ we have $\sum_{i \in C} d_i^*(\mathbf{y}) < 1 + \delta\left(\sum_{i \notin C} \pi_i\right) d_h^*(\mathbf{y})$. This is because an alternative in Δ_m that allocates amount equal to their demand to each of the legislators in $C \backslash \{h\}$ as well as an amount $x > 0$ such that

[15] Result holds a fortiori if $l > m$.

$d_h^*(\mathbf{y}) = \dfrac{x}{1-\delta\left(\sum_{i\notin C}\pi_i\right)}$ to h, is such that the demand of all m is satisfied. Suppose the contrary, *i.e.* $\sum_{i\in C} d_i^*(\mathbf{y}) \geq 1+\delta\left(\sum_{i\notin C}\pi_i\right) d_h^*(\mathbf{y})$. Consider two cases: For $l>1$, from step (11) we have $\sum_{i=1}^{l-1}\left[U_i^*(\mathbf{y})-\delta\overline{v}_i\right] \geq \sum_{i=1}^{l-1} u(y_i) - \delta\left(\sum_{i=1}^{l-1}\pi_i\right) D$. Since $\sum_{i=1}^{n}\left[U_i^*(\mathbf{y})-\delta\overline{v}_i\right]=1$, we must then have $\sum_{i=1}^{l-1}\left[U_i^*(\mathbf{y})-\delta\overline{v}_i\right]+\sum_{i\in C} d_i^*(\mathbf{y})<1 \Rightarrow \sum_{i=1}^{l-1} u(y_i)-\delta\left(\sum_{i=1}^{l-1}\pi_i\right) D+ \delta\left(\sum_{i\notin C}\pi_i\right) d_h^*(\mathbf{y}) < 0 \Longrightarrow \left(\sum_{i\notin C}\pi_i\right) d_h^*(\mathbf{y}) < \left(\sum_{i=1}^{l-1}\pi_i\right) D$, which is false by the definition of D, and the fact that $\left(\sum_{i\notin C}\pi_i\right) \geq \left(\sum_{i=1}^{l-1}\pi_i\right)$, and so is the original assumption $\sum_{i\in C} d_i^*(\mathbf{y}) \geq 1+\delta\left(\sum_{i\notin C}\pi_i\right) d_h^*(\mathbf{y})$. For $l=1$ from $\sum_{i=1}^{n}\left[U_i^*(\mathbf{y})-\delta\overline{v}_i\right]=1$ we obtain $\left(\sum_{i\notin C}\pi_i\right) d_h^*(\mathbf{y})<0$, again a contradiction. If risk neutrality is small enough $[u(x)-x<\varepsilon]$, we have $\sum_{i=1}^{n}\left[U_i^*(\mathbf{y})-\delta\overline{v}_i\right] < 1+n\varepsilon+\delta m\varepsilon+\delta^2 2\varepsilon$ [since absorption in $\Delta_{2\kappa}$ occurs in at least three periods]. Since the inequalities above are strict (and hold with wider margins if probabilities of recognition are more symmetric) (B) holds for small enough ε as well. ■

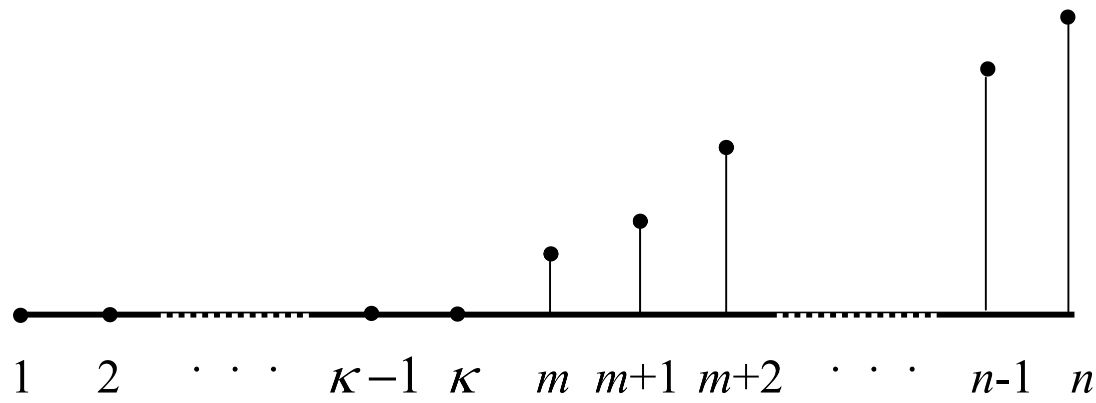
TABLE 1: Example of Conjectured Equilibrium Path (*n*=5)

Legislator	*Period*					
	t	*t+1*	*t+2*	*t+3*	*t+4*	*t+5*
1	x_1	y_1	0	0	0	0
2	x_2	y_2	0	1	0	0
3	x_3	y_3	z	0	0	1
4	x_4	0	0	0	0	0
5	x_5	0	1-z	0	1	0

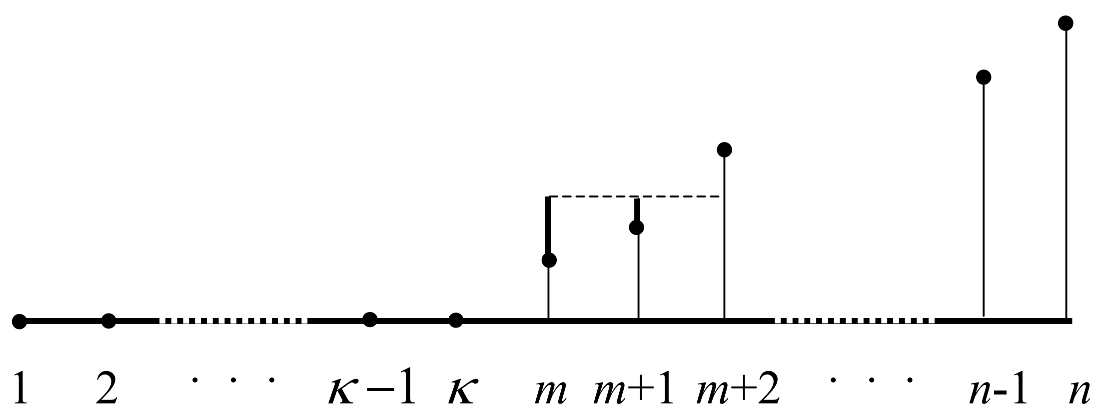
key: Proposers form minimum winning coalitions and legislators that have zero under status quo vote *yes* if proposer allocates them zero. In at most three periods proposer extracts the whole dollar. In the example above, proposer in period t+1 is any one of legislators 1, 2, or 3. In period t+2 proposer is legislator 5 and allocates an amount to 3 to form a majority with 3, and 4. Proposers in subsequent periods are 2, 5, and 3 respectively.

FIGURE 1: Optimal Proposal When $\mathbf{s} \in \Delta_{\kappa}$

(a) Status quo allocation

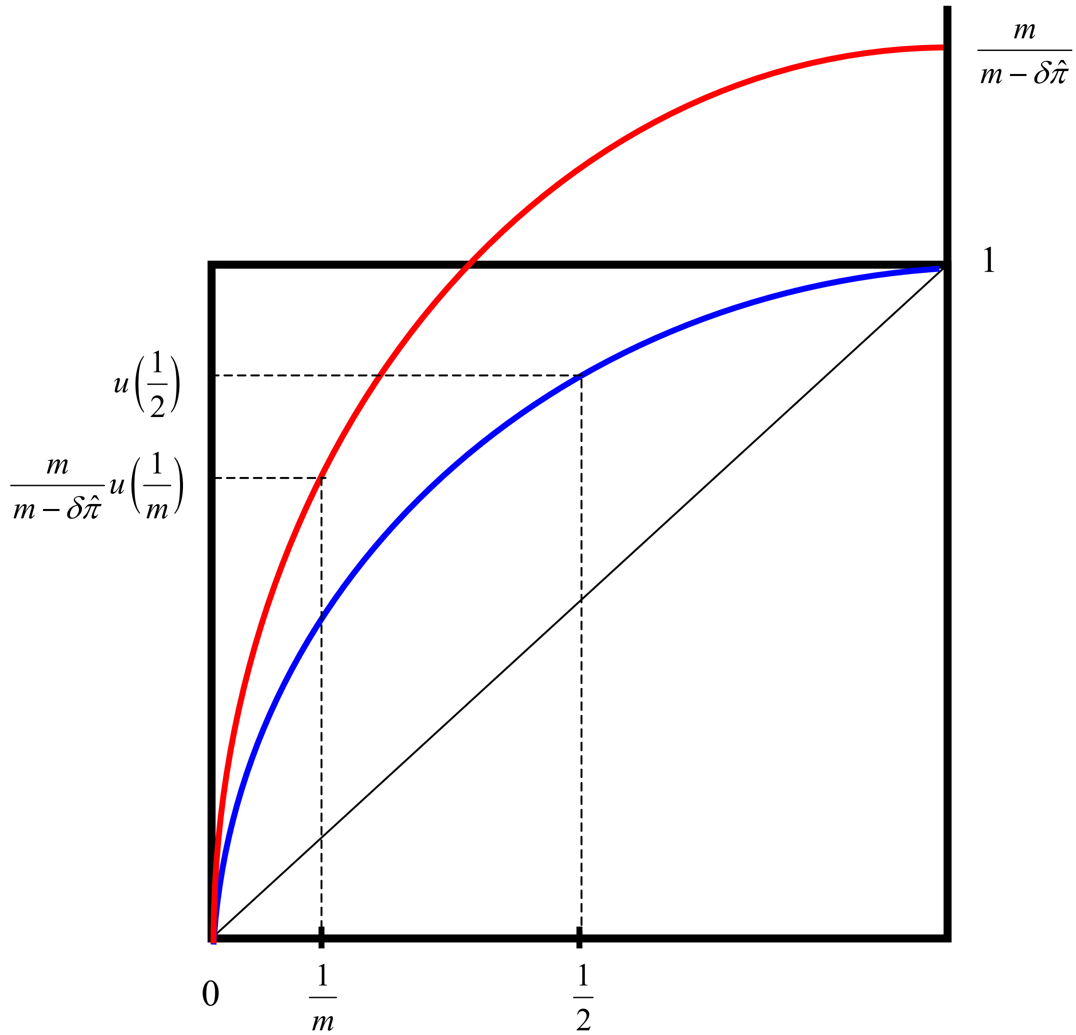


(b) Equilibrium demands



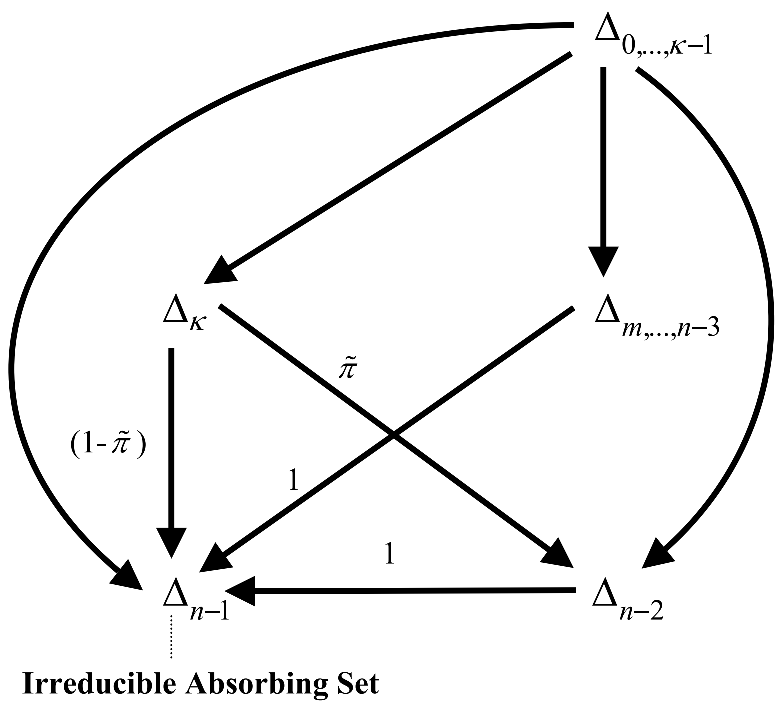
key: If legislator m is always chosen as coalition partner by 1, …, κ, she demands an amount that exceeds the allocation of legislator m+1. In equilibrium, legislators m and m+1 are chosen with positive probability and demand equal amounts. Legislators m+2, …, n demand their allocation under status quo.

FIGURE 2: Necessary Equilibrium Conditions & Risk Aversion



key: If $\frac{m}{m-\delta\hat{\pi}}u\left(\frac{1}{m}\right)>u\left(\frac{1}{2}\right)$, there exist states $\mathbf{s}\in\Delta_\kappa$ for which a proposer with zero share of the dollar is better off allocating $\frac{1}{m}$ to m legislators including herself, as opposed to allocating a positive amount to a single legislator with positive share of the dollar under the state, ***s***.

FIGURE 3: Equilibrium Induced Markov Process



key: From any initial allocation, outcomes are absorbed in the set where a single player (the proposer) obtains the whole dollar in at most three periods.