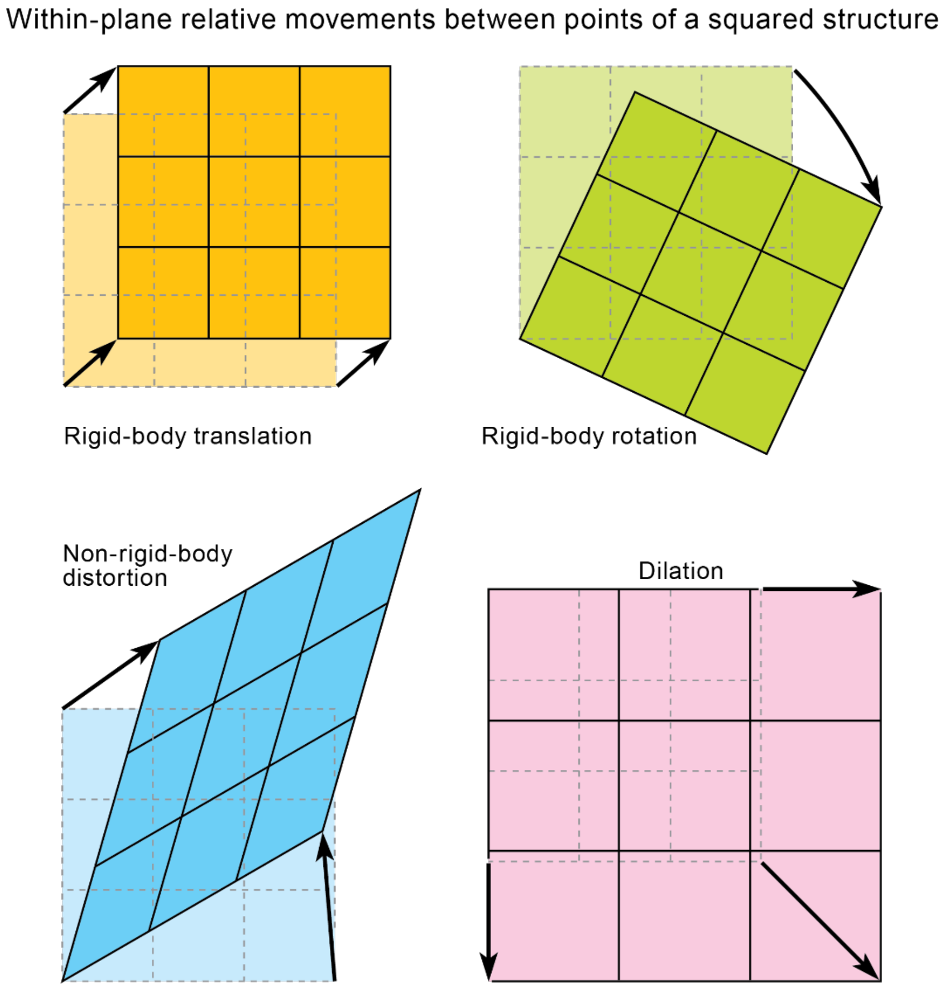
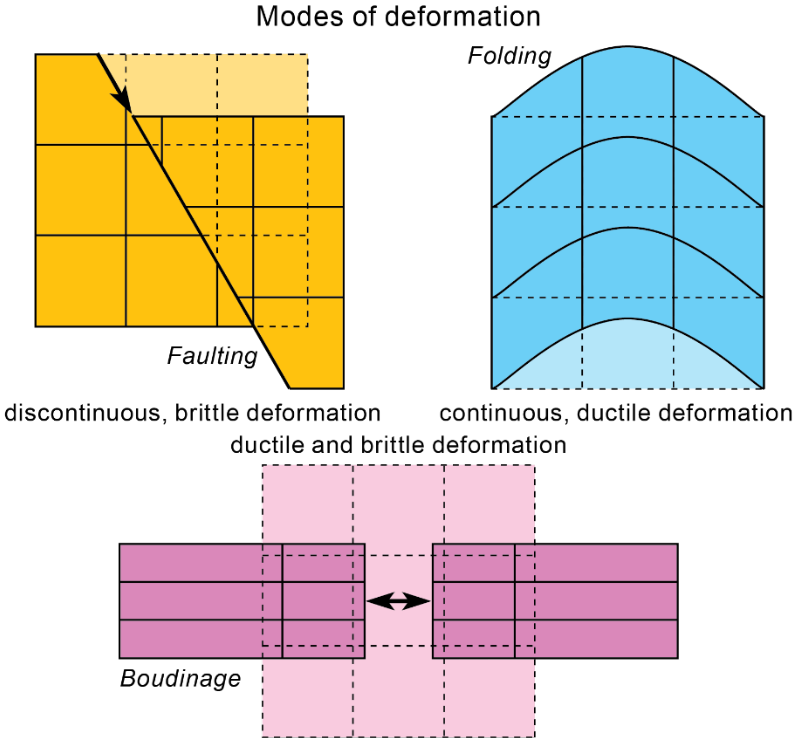
MECHANICAL ASPECTS OF DEFORMATION

Mechanics deals with the effects of **forces** on **bodies**. A solid body subjected to external forces tends to change its position, displacement, or shape. During **rigid body deformation**, rocks are translated and/or rotated while their original size and shapes are preserved.



If a body absorbs some or all the forces acting on it instead of being moved, the body becomes **stressed**. The forces then cause particle displacements, resulting in the body's shape changing and becoming **deformed**. **Strain** refers to the change in shape or non-rigid body deformation of a rock resulting from the application of stresses.



On Earth, the most significant forces are due to gravity and the relative motions of large rock masses in the crust and the mantle. Other possible forces are usually small or act only for short periods, so that no significant strain results.

Movement-related forces act for relatively long times. Structural geology is concerned with the permanent deformation, or **failure**, that produces **structures** such as folds and faults in rocks. If a rock fractures and loses cohesion, it is considered **brittle**. If the rock had deformed without losing cohesion and retained its intricate shapes when the forces ceased acting, the rock would display a permanent strain and be considered **ductile**. The **behaviour** of the rocks, that is, whether they deform permanently or not and whether any deformation is predominantly by folding, by faulting, or by yet other modes, is influenced by the interplay of several physical and chemical factors. Thus, a thorough understanding of the deformation process is important.

The primary purpose of this lecture is to examine some of these factors and gain a physical understanding of how rocks deform in nature. The description of any deformation process involves specifying the loads applied, which is the goal of a **dynamic analysis**. Indeed, the propensity for rock deformation can be estimated using easily measured material properties, such as the flow stress as a function of strain, strain rate, and temperature, which apply to many rocks. Hence, this discussion defines the concepts of stress, strain, rheology, and equations of motion. Nobody can see stresses directly; one can only infer them from the results of deformation. This lecture will begin with this topic and introduce the vector calculus it involves.

Physical definitions

Continuous medium

Rocks are complex assemblages of crystals, grains, fluids, and other materials whose properties and physical parameters vary continuously. Continuous variation implies that these parameters have spatial derivatives. It is, therefore, necessary to consider infinitely small volumes of material in which physical properties are the same everywhere. This is a continuous medium that models real materials without considering their fine (e.g. atomic) structure. The mechanical discussion that follows considers rocks as continuous media.

Newton's axioms: Laws of motion

Newton's axioms state the conditions under which a body moves in response to an external quantity, the force, and a characteristic of the body: its mass (i.e., the amount of material in the body). In **dynamics**, only the driving forces of the movements are considered. Since deformation is defined as the relative movement of points, Newton's three laws of motion are considered fundamental axioms.

Law 1, (inertia principle)

A body remains in a state of rest or in uniform motion in a straight line unless new forces act upon it, compelling it to change that state or motion.
The rate of such a "free-moving" body is constant in terms of magnitude and direction.

Law 2, (action principle)

The change of motion is proportional to the force impressed and is in the same direction as the line of the impressed force.

Law 3, (reaction principle)

To every action F there is always opposed an equal reaction $F_R = -F$; or, the mutual actions of two bodies on each other are always equal and directed to opposite parts.
For example, a falling rock exerts the same force on the Earth as the Earth exerts on the rock.

Dimension / Quantity

Mechanical properties of a material are expressed in terms of the three independent, physical dimensions (i.e., measurable parameters) **length** [L], **mass** [M], and **time** [T], [ ] meaning "has the quantity of". Other dimensions, such as electrical charge [Q] and temperature [θ], are **derived dimensions**.
A quantity is the numerically scaled magnitude of a physical dimension. Quantities are conventionally expressed in the *Système international d'unités* (SI units). These units are meter (m), kilogram (kg), and second (s) for length, mass, and time, respectively.
- One meter is the distance that light travels in a vacuum during 1/299'792'458 of a second.
- One kilogram is the mass of the International Prototype Kilogram stored in a vault at the *Bureau International des Poids et Mesures* in Sèvres (France). It is almost exactly equal to the mass of one liter of water.
- One second is "the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the Caesium 133 atom" (after the Certificate in Investment Performance Measurement) at a temperature of 0 K.

Force

A force is what influences or tends to change the motion of a body.

Mathematical expression

A force possesses both **magnitude** and **direction**. Therefore, a force $\vec{F}$ is a vector quantity that follows the rules of vector algebra. Conventionally, an arrow in a given coordinate system represents it.
- The length of the line specifies the amount of the force (e.g. how strong a push is).
- The orientation of the line specifies its direction of action (i.e., the direction in which the push is directed).
- An arrow pointing in the direction of acceleration indicates the sense of direction.

The action principle states that a force $\vec{F}$ acting on a body of mass m will accelerate the body in the direction of the force. The acceleration $\vec{a}$ is inversely proportional to the mass m and directly proportional to the acting force:

$$\vec{a} = \vec{F}/m \Leftrightarrow \vec{F} = m.\vec{a}$$

This relationship is also written:

$$\vec{F} = \frac{\overrightarrow{mv}}{t} = \frac{d(\overrightarrow{mv})}{dt}$$

where $\overrightarrow{mv}$ is the product of mass and velocity, i.e. the **momentum**, and t is time.

The most familiar force known to us is weight, which, by definition, is the force experienced by a mass (the product of volume and density) in the direction of gravity's acceleration and, hence, normal to the Earth's surface.
As with any vector quantity, a force may be **resolved** into several **components** acting in different directions, according to the parallelogram rule of vector analysis. For example, any force $\vec{F}$ can be resolved in three components labelled $\vec{F}_x$, $\vec{F}_y$ and $\vec{F}_z$, parallel to the coordinate axes x, y and z, respectively. This is conveniently written in a column form:

$$F = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$$

Orientation

To define orientations, one applies the concept from vector analysis that two vectors can define a plane. This first definition is extremely useful in 3D applications, as the product of these two vectors is a vector perpendicular to both, i.e., perpendicular to the plane that contains them. The vector product is also a unit vector, as its magnitude is 1. The practical application is the standard Cartesian coordinate system, for which $\vec{\imath}$ is the unit vector along the x-axis (thus orthogonal to the yz plane), $\vec{\jmath}$ is the unit vector aligned with the y axis (perpendicular to the xz plane), and $\vec{k}$ is the unit vector along the z-axis (perpendicular to the xy plane). Expressing fully the force vector is therefore:

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = F_x.\vec{\imath} + F_y.\vec{\jmath} + F_z.\vec{k} = F_x.\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + F_y.\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + F_z.\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or

$$\vec{F} = \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix} = F_x.\vec{e}_x + F_y.\vec{e}_y + F_z.\vec{e}_z$$

with $\vec{e}_i$ the unit vector along the i-axis.
When one writes the three vector components, i.e. how long each vector is along each of the 3 axes, one omits the unit vectors, and coordinates are simplified to the coefficients of the $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ parts of the equation. The vector entity, defined by three numbers and a coordinate system, is mathematically described as a first-order tensor.
Forces and force components are added as vectors.

Dimension

The unit and dimension of a force are defined from the second Newton's law: $\vec{F} = m.\vec{a}$. The dimension has the form:

$$[F] = [M.L.T^{-2}]$$

The mass is a **scalar** quantity, i.e. it requires only one number to define it. Its unit is the kilogram (1kg). Mathematically, a scalar is an entity also referred to as a zero-order **tensor**.
Acceleration needs a coordinate system to be defined.

The Newton and the dyne are the basic units of force (1 N = force required to impart an acceleration of 1 m/s^{-2} to a body of 1 kg:

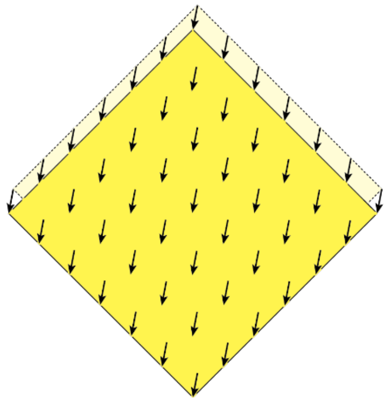
$$1\text{N} = 1\text{kg}.1\text{m}.1\text{s}^{-2} = 10^5\text{dynes}$$

Surface - body forces

Body forces

Body forces result from the action of distant, external forces (such as gravity, the electromagnetic field, etc.) on every particle of the body; for example, gravity acts on every atom of a pen, producing its weight. Body forces are consequently proportional to the mass and, hence, to the volume of the body.

Body force
(forces that act on each point of a body)

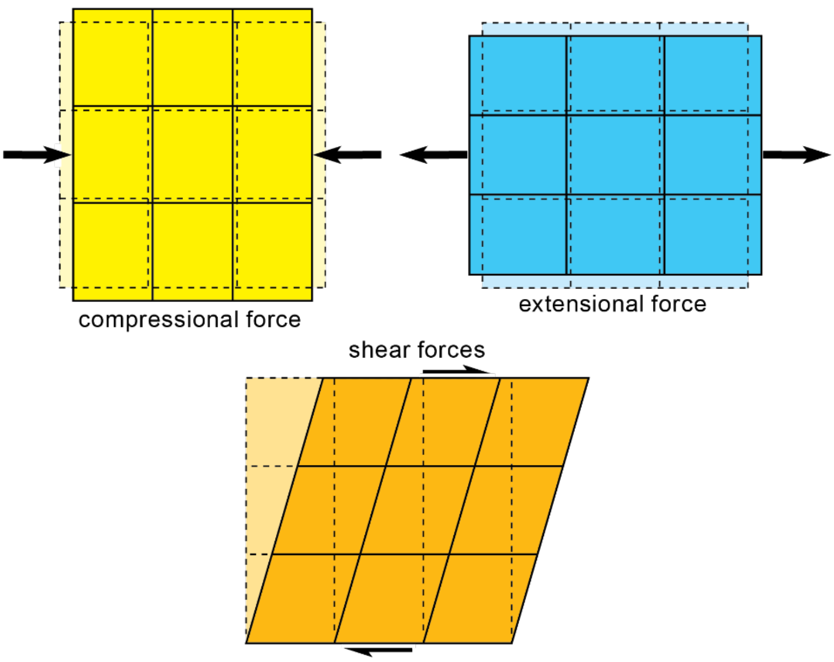


In purely mechanical systems, the body forces are of two kinds: those due to gravity and those due to inertia.

Surface forces

Surface forces (or **applied forces**) act on the external boundaries of a body as well as on any imaginary or real surface within this body.

Orientation of surface forces
(forces that act on the boundaries of a body)



No concrete physical surface or visible material boundary is required. Surface forces, such as friction, are proportional to the size of the area upon which they act. Surface forces can result from the action of the body on itself, such as the tension in a stretched rubber band. Surface forces are usually generated beyond the considered body. They are transmitted to it through the entire mechanically continuous region that connects it to the point where the force is exerted. For example, the action of pushing on its extremity and displacing a pen. In geology, tectonic forces can be transmitted through the plate from its boundaries.

Ratio of the body forces to the surface forces

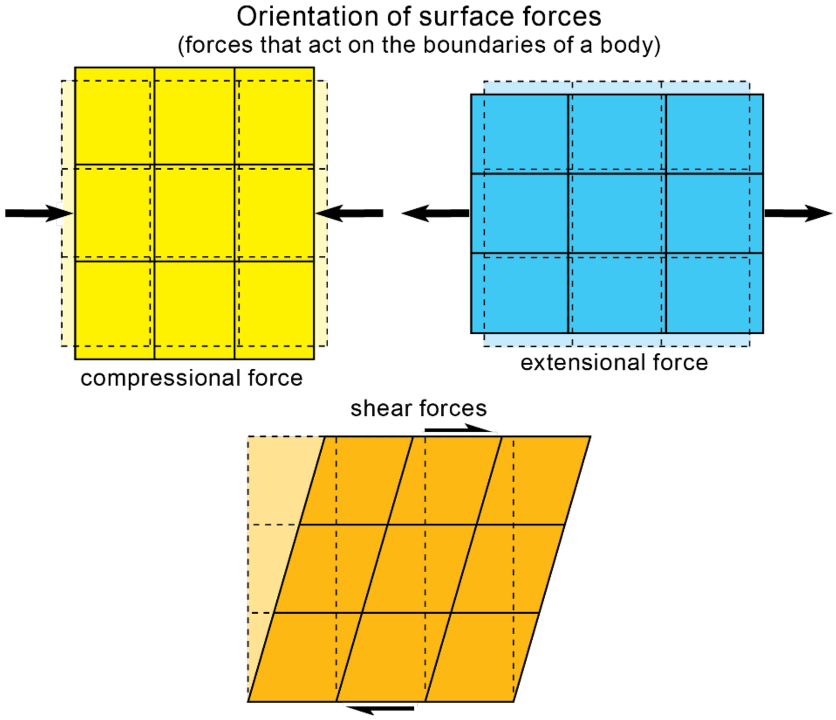
Since gravitational forces are directly proportional to mass, the weight of an overlying column of rocks constitutes a significant force on rocks at depth.
In general, each element of mass is in a state of dynamic equilibrium, which means that the sum of body forces is equal and opposite to the sum of surface forces. If $\mathrm{d}\ell$ is the characteristic length of a small body element, the ratio of body forces to surface forces is:

$$\frac{\text{Body forces}}{\text{Surface forces}} = \mathrm{K}\frac{(\mathrm{d}\ell)^3}{(\mathrm{d}\ell)^2}$$

which tends to be zero as $\mathrm{d}\ell$ tends to zero. The difference in power implies that the magnitude of body forces diminishes more rapidly (considerably for large $\mathrm{d}\ell$) than that of surface forces. Consequently, if the volume element is small, the body forces in equilibrium with themselves may be neglected. The scaling between body and surface forces has, for example, important corollaries in biological engineering; the strength of a bone is proportional to its cross-sectional area, but the weight of the body is proportional to its volume. The bones of larger animals, therefore, have greater diameter-to-length ratios to hold relatively larger weights. Such scaling relations also control mechanical deformation processes. For example, George Gabriel Stokes derived a solution in 1851 for the velocity of a sphere falling (or rising) in a viscous fluid with a lower (or higher) density. This velocity increases with the square of the radius of the sphere because the buoyancy forces increase relatively more than the friction forces acting on the sphere surface against its motion. Similar relationships between buoyancy and friction forces have important tectonic implications for the rise of diapirs and the sinking of lithospheric plates into the mantle.

Directed forces

Directed forces act in particular directions. In geology:
- **Compression** is a pair of in-line forces that tends to compress bodies;
- **Tension** is a pair of in-line forces that tends to pull bodies apart;
- **Shear** refers to coupled forces acting in opposite directions in the same plane but not along the same line;
- **Torsion** is a twisting force.



Normal and shear components

A force $\vec{F}$ acting on a plane is generally oblique to the surface and may be resolved into vector components, acting perpendicular and parallel to the plane.

$$\vec{F} = \vec{F}_N + \vec{F}_S$$

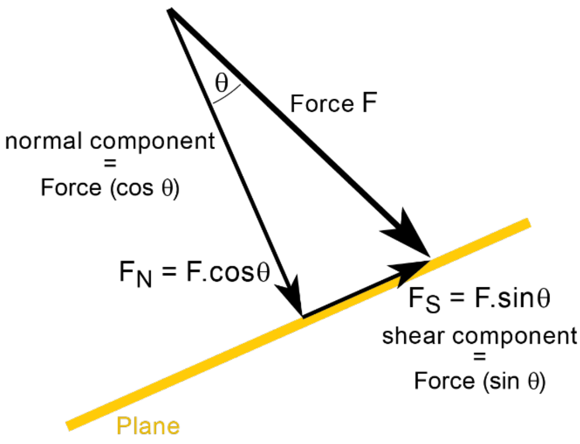
$\vec{F}_N$ and $\vec{F}_S$ are the **normal** and **shear** forces, respectively. The shear component facilitates slip on the plane while the normal component tends to prevent it, pressing both sides of the plane towards each other.

In two dimensions, $\vec{F}$, $\vec{F}_N$ and $\vec{F}_S$ are coplanar; the two perpendicular components are defined according to the right angle trigonometry as:

$$F_N = F\cos\theta$$
$$F_S = F\sin\theta$$

with θ the angle between the applied force and the normal to the considered plane (line in 2D). The magnitude is obtained using the Pythagoras' Theorem:

$$F^2 = F_N^2 + F_S^2$$

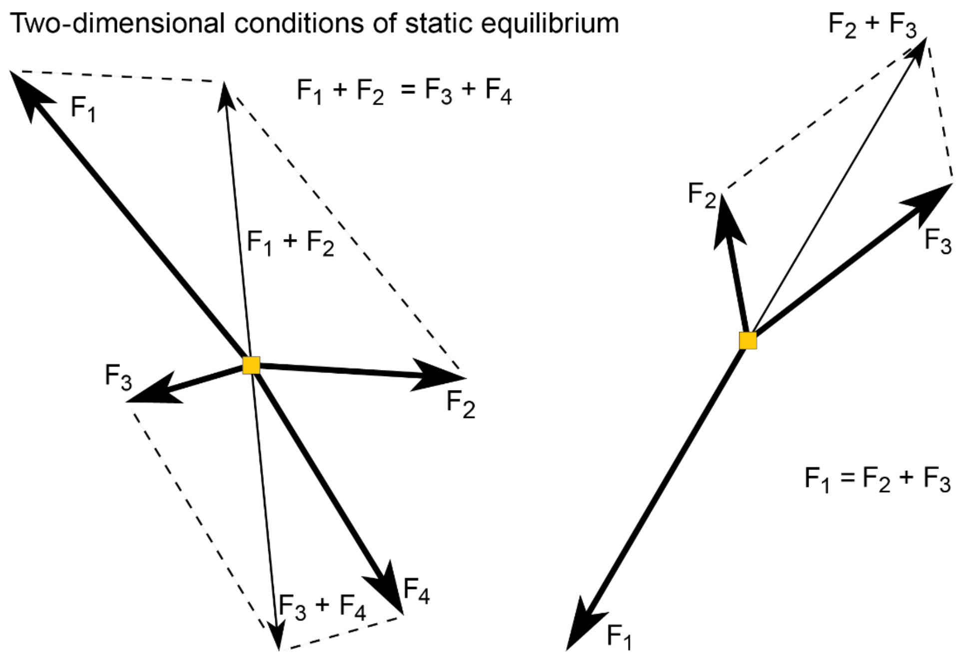


Two-dimensional vectorial decomposition of a force acting oblique to a plane into its normal- and shear components

These equations demonstrate that the key to determining the component magnitudes is to know (i) the magnitude of the applied force vector and (ii) the angle it forms with the plane.

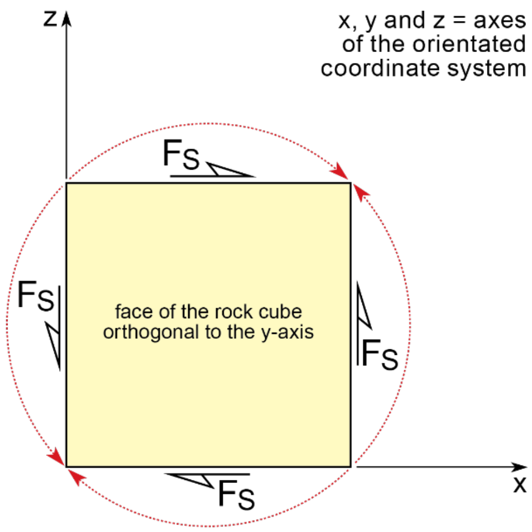
Action / reaction; static equilibrium

Imagine a cube of rock within a large volume of rock assumed to be a continuous material. The six faces of the imagined cube are pressed against adjacent parts of the rock, while corresponding reactions occur within the cube. Newton's reaction principle, stating that forces occur in pairs that are equal in magnitude but opposite in direction, expresses this situation. In addition, each atom within the cube is acted on by gravity, but each atom outside the cube is also. Therefore, the general body force, which is equal everywhere, can be, in a first approach, considered to be absent.



Two opposed forces of same magnitude act on an element of mass

In **static equilibrium**, the considered cube of rock is neither moving nor deforming. The system of forces is closed, and the sum of all forces in all directions equals zero. Static equilibrium is the situation treated to understand natural geological forces.



Equality of shear forces in static equilibrium
For the torque on a volume element in a continuum to be null (no angular acceleration), the shear forces acting on opposite faces and acting in opposite directions (clockwise / anticlockwise) must compensate each other

In that case, forces on opposite faces cancel out, and there is no net couple that would rotate the cube. This again requires that forces on opposite faces be equal in magnitude and opposite in sense. The shear forces on opposite faces must also be balanced. To simplify the argument, we take the cube edges as the principal axes of a three-dimensional coordinate system. Then, the shear component is resolved into two shear components parallel to the face edges.

Stress in a continuous medium

Stress is what tends to deform a body.

Definition

The magnitudes of the forces acting on the external faces of the cube depend on the areas of these faces: the larger the cube, the larger the force required to produce a change in shape or movement. The situation is complicated by variations in magnitude and direction of force from point to point over each cube face. Therefore, it is convenient to have a measure of the deforming forces that is independent of the size of the cube considered. This freedom in the calculation is procured by imagining the cube to shrink to a cubic point whose infinitely small faces have area $A = 1$.
The significance of the area on which a force is applied is intuitively known to all of us. The feet sink when walking on snow, to a lesser extent if one has snowshoes, and one can even slip on skis. The force (weight of the person) acting on the snow is the same, but increasing the contact area reduces the stress on the snow. This indicates that stress, rather than force, controls the deformation of materials (in this case, snow). Therefore, one needs to work with stress to investigate the deformation of rocks.

Traction

The **traction** T represents the force intensity with respect to the surface area on which it is applied. If the force F is uniformly distributed over a large area, then:

$$T=F/A$$

If the force varies in direction and intensity over the area, then traction should be defined only at a point considered as an infinitesimal area. Using the imaginary cubic point, then traction is formally defined as the **force (F) per unit area** applied in a particular direction at a given location on the cube.

A more precise definition of the traction at a point is given by the limiting ratio of force $\Delta\vec{F}$ to the area ΔA as the face area is allowed to shrink and approaches zero (Cauchy's principle).

$$\vec{T} = \lim_{\Delta A \to 0}\left(\frac{\Delta\vec{F}}{\Delta A}\right) = \frac{dF}{dA}$$

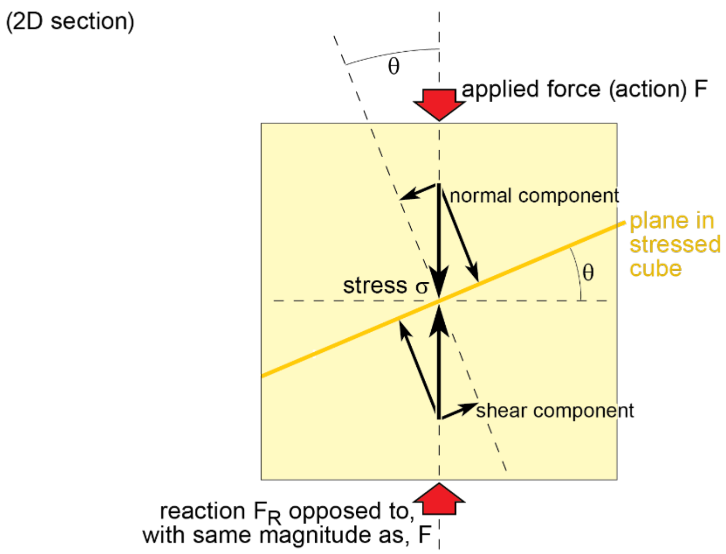
In this equation, $\Delta\vec{F}$ is a vector quantity defined by three quantities:
- its magnitude;
- its orientation;
- the orientation of the plane on which it is applied, which is defined by the normal unit vector $\vec{\boldsymbol{n}}$.

$$\vec{F} = \vec{T}.A.\vec{n}$$

This definition contains two directional components: one for force and one for plane orientation. It indicates that traction is a **bound vector** that may vary from point to point on any given plane and be infinitely differentiable on the infinite number of planes that pass through any given point. Traction is, therefore, always expressed with reference to a particular plane.

Stress

Assuming mechanical equilibrium (law of motion 3, reaction principle), if traction is applied to the external surface of a body, then it sets up internal tractions within the body. The same equation that defines external traction also defines internal traction; hence, there are equal but opposite tractions on both sides of the contact cubic point. This pair of balanced tractions is the **stress**. Stress is applied to any point of a body, like spring tension: there are equal and opposite forces on the other (hidden) three faces of the cubic point. Since stress comprises both the action and the reaction, stress is defined as a **pair of equal and opposite forces** acting on the unit area. Stress is transmitted through the material by the interatomic force field. The body is then in a **state of stress.**



Stress (a pair of equal and opposite vectors) induced at a point of an inclined plane within a body subjected to uniaxial compression

Dimension

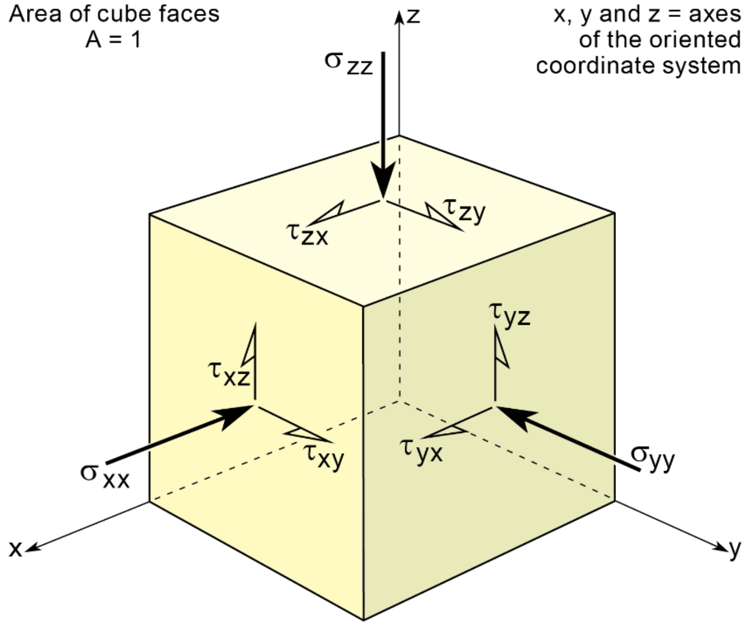
Stress, as **pressure**, includes the physical dimensions of force and those of the area on which the force is applied:

$$[M{*}LT^{-2}]/[L^2] = [\text{Mass} * \text{Length}^{-1} * \text{Time}^{-2}]$$

The unit is the Pascal (1 Pa = 1 Newton.m^{-2}, remembering that 1N = 1 kilogram meter per square second: 1 kg.m^{-1}.s^{-2}) and Bars with 1 Bar = 1b = 10^5 Pa ~ 1 Atmosphere. Geologists more commonly use 1 megapascal (MPa) = 10^6 Pa. A useful number to remember for discussion with metamorphic petrologists in particular, is 1 kb = 100 MPa. 100 MPa is approximately the lithostatic pressure at the bottom of a rock column of 4 km height and with a density of 2600 kg.m^{-3} (2600 kg.m^{-3} x 4000 m x 9.81 m.s^{-2} = 102 MPa, see section "Terminology for state of stress").

Stress components

With an infinitely small cube, body forces are negligible compared to surface forces. Hence, body forces are in equilibrium with themselves, and one can consider the **state of stress at a point** (the infinitely small cube) within the body. Since stress cannot be defined without specifying the plane upon which the stress acts, both the direction of the force and the orientation of the faces of the cube must be considered.



Cartesian components of stress at a (cubic) point resolved into normal- (σ) and shear- (τ) stress components
Note that only stresses acting on the three visible sides of the cube are drawn. Stresses on the opposite, hidden sides of the cube should also be represented and all stresses should be pointing to the central point within the cube.

Forces (and traction vectors) on each of a cube's faces are decomposed into three mutually orthogonal components, one normal to the face (the normal force) and two parallel to the face (the shear forces). Like forces, stresses acting on an infinitely small cube whose faces are unit areas can be decomposed into three **normal stresses** perpendicular to the faces and three **shear stresses** parallel to each face; each shear stress is parallel to one of the coordinate directions contained in the face plane.

- The normal stress, transmitted perpendicular to a surface, is given the symbol σ.
- The shear stresses, transmitted parallel to a surface, have the symbol τ but σ is common notation in the literature.

<u>Exercise; graphic representation to be done with Excel</u>

** Draw a square ABCD and a diagonal surface on it.*
** Draw a vertical force F_1 that acts on this surface.*
** Write equations that express the normal and shear components on this surface.*
** Show that the highest shear stress is obtained for an angle θ of 45° between the surface and F_1.*

** Represent graphically variations of the normal force F_N and the shear force F_S on the surface as a function of the angle θ.*

Stress at a 'point' in a continuous medium

The **state of stress** at a point is three-dimensional. It is convenient to use the edges of the infinitesimally small cube (the "shape" of the point) as a system of Cartesian coordinates (x_1, x_2, x_3). Employing the symbol σ_{ij} to denote the component of stress that acts on the pair of faces normal to x_i (thus identifying the plane orientation) and in the direction of x_j (thus defining the direction of traction), one resolves the stresses that act on the faces of the cube normal to x_1 into:

σ_{11} the normal stress component, perpendicular to the faces normal to x_1(or x).

τ_{12} and τ_{13} the two shear components within the paired faces normal to x_1, each acting parallel to one of the other coordinates axes x_2 and x_3 (or y and z), respectively.

For each pair of faces, there is one face for which the inward-directed normal stress, taken here as positive, is opposite to the normal stress acting on the other face. The same procedure applies to the faces normal to x_2 and x_3 (or y and z), so that a total of nine stress components is obtained for the three pair of faces:

Pair of faces normal to x_1:	σ_{11}	τ_{12}	τ_{13}		σ_{xx}	τ_{xy}	τ_{xz}
Pair of face normal to x_2:	σ_{22}	τ_{21}	τ_{23}	or written as	σ_{yy}	τ_{yx}	τ_{yz}
Pair of face normal to x_3:	σ_{33}	τ_{31}	τ_{32}		σ_{zz}	τ_{zx}	τ_{zy}

These are written so that components in a row act on a plane and components in a column act in the same direction. Using the symbol σ instead of τ yields the following ordered array:

$$\begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix}$$

This geometrical arrangement represents the original set of coefficients that form the **stress matrix**:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{1}$$

A matrix that has the same number of rows and columns is a **square matrix**. One may collectively call this matrix of coefficients $\boldsymbol{\sigma}$ or σ_{ij}, identifying its elements in a simple form as:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

This grouping of the nine stress components is the **stress tensor**.

Reminder: Mathematical definitions; what are we talking about?

Scalar: a quantity with magnitude only (i.e., a real number, such as mass, temperature, or time).

Vector: A geometrical object with magnitude and direction (e.g., force, velocity, acceleration).

Tensor: a mathematical structure with magnitude and two directions (two vectors), one (a unit vector) specifying a plane of action (e.g. permeability, strain, stress).

The stress tensor, which represents all possible traction vectors at a point with no dependence on the orientation of the plane (unit normal vector), fully describes the state of stress at a point. More specifically, it is a symmetric tensor since the six off-diagonal components are interchangeable; it is a second order tensor since it is associated with two directions. Accordingly, stress components have 2 subscripts, where indifferently and independently i = 1, 2, 3 and j = 1, 2, 3. The subscripts i and j refer to the row and column location of the element, respectively. The diagonal components $\sigma_{i=j}$ are the normal stresses and the off-diagonal components $\sigma_{i\neq j}$ are the shear stresses.
If the elemental cube does not rotate (i.e. postulating equilibrium condition and no body forces), shear stresses on mutually perpendicular planes of the cube are equal: three of the shear components counteract and balance the other three, i.e. the rotating moments about each of the axes, the **torques** read across the diagonal of the square matrix, are zero:

$$\begin{matrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{matrix}$$

Reminder
Torque is the product of the force vector and the perpendicular distance between the center of mass and the point of application of the force.

Since $\sigma_{ij} = \sigma_{ji}$ (i.e. subscripts for shear stress magnitudes are commutative), the **symmetrical** stress matrix is:

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

Its nine components reduce to six truly independent stress components acting on any arbitrary infinitesimal element in a stressed body:

normal stresses	σ_{11}	σ_{22}	σ_{33}
shear stresses	σ_{12}	σ_{23}	σ_{31}

Therefore, for an arbitrarily chosen set of orthogonal axes x, y and z, six independent quantities are necessary to specify completely the state of stress at a point, i.e. for every surface element leading through the point.
The cube representation helps emphasize an important difference between stress and forces. A directed force may be acting in a certain direction (say, towards the left), but this statement lacks meaning when applied to internal stresses. A stress component acting upon one side of a surface element exists only together with a component of equal intensity but opposite direction, acting on the other side. This is true for both normal and shear stresses. Hence, stress may exist in a vertical direction but not in the direction of up or down.

Principal stresses

Even if six independent stress magnitudes and unconstrained orientations simplify the stress tensor, the formulation remains somewhat cumbersome to employ. Fortunately, this situation can be considerably simplified. It is always possible, at any point in a homogeneous stress field, to find three mutually orthogonal planes intersecting at the point and oriented such that all shear stresses vanish to zero. Thus:

$$\tau_{12} = \tau_{23} = \tau_{31} = 0$$

In this case, there remain only the normal components of stress and:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \quad \text{becomes} \quad \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

These three no-shear-stress planes are the **principal planes of stress** and intersect in three mutually perpendicular lines, known as the principal axes of stress at the considered point. The stresses acting along these three axes are the **principal stresses** σ_{11}, σ_{22} and σ_{33} denoted σ_1, σ_2 and σ_3 to avoid repetitive subscripts, with the convention that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, the **maximum, intermediate**, and **minimum** principal stresses, respectively. In other words, the principal stresses are the normal stresses that act on planes of zero shear stresses. They coincide with the principal axes of the stress ellipsoid, which will be defined further.

<u>Attention! Sign convention:</u>

In physics and engineering, tensile normal stress, which tends to pull material particles apart, is considered positive; conversely, compressive normal stress, which tends to push the material particles together, is negative. In geosciences, it is customary to define compression as positive and tension as negative because natural stresses are typically compressional, even in areas experiencing extension. For example, in a non-tectonic environment, the stress at any depth within the Earth is generated by the overburden. It is a compressive vertical stress that induces a compressive horizontal stress. Even at the Earth's surface, the maximum compressive stress is equal to the atmospheric pressure. Shear stresses are positive anticlockwise.

If the magnitudes and orientations of the three principal stresses at any point are known, the components of normal and shear stress on any plane through that point can be computed. The state of stress at a point may, therefore, be completely characterized by specifying the magnitude of these three principal stresses and their respective directions. The six independent stress components are needed only when the faces of the reference cube are not parallel to the principal planes of stress.

Terminology for states of stress

Some particular stress states are:

$\sigma_1 = \sigma_2 = 0$;	$\sigma_3 < 0$	Uniaxial tension
$\sigma_2 = \sigma_3 = 0$;	$\sigma_1 > 0$	Uniaxial compression
$\sigma_2 = 0$		Biaxial (plane) stress
$\sigma_1 > \sigma_2 > \sigma_3$		General, triaxial stress
$\sigma_1 = \sigma_2 = \sigma_3 = p$		**hydrostatic** state of **stress**; all shear stresses are zero.

If $p < 0$ (**tensile**) the stress state is referred to as a **hydrostatic tension**. Hydrostatic stresses will cause volume changes but not shape changes in a material.

In geology, the **lithostatic pressure** is often used to describe the hydrostatic pressure generated at a depth *z* below the ground surface due solely to the weight of rocks, of mean density ρ, in the column. Naturally, this is equal to ρgz where g is the acceleration due to gravity. Such a statement, however, requires some qualification because it assumes that the stress state at depth z has become truly hydrostatic due to the relaxation of all shearing stresses by some creep process. If the stress state has not been allowed to become hydrostatic, and one talks about the stress state due solely to a pile of rocks of height z, then this is usually taken to be:

$$\sigma_1 \approx \int_0^z \rho g.dz$$

$$\sigma_2 = \sigma_3 = \left[\nu/(1-\nu)\right]\sigma_1$$

where ν is **Poisson's ratio**.

Mean stress

The **mean stress** $\bar{\sigma}$ or **hydrostatic stress component** p (also called dynamic pressure) is the arithmetic average of the principal stresses:

$$\bar{\sigma} = p = \left(\sigma_{11} + \sigma_{22} + \sigma_{33}\right)/3 = \sigma_{ii}/3$$

This pressure is independent of the coordinate system; it has equal magnitudes in all directions.

Reminder, fundamental terminology:
In mathematics, the sum of the diagonal components of a tensor, which does not change with rotation of the coordinate system, is the **first invariant**.

In the Earth, the mean stress typically increases by ca. 30 MPa/km (ca 3kbar/10km). The mean stress thus specifies the average level of normal stress acting on all potential fault planes, which governs the frictional resistance to slip on fault planes. Otherwise, the mean stress may only produce a change in volume, either reducing it if the mean stress is compressive or expanding it if it is tensile.

Deviatoric stress

Observable strain results from distortion, while it is difficult to measure volume changes in rocks. Therefore, strain is typically related to the distance between stress and the isotropic state. The **deviatoric stress** expresses this difference by subtracting the mean stress from the stress tensor. Considering that any general state of stress is the sum of the hydrostatic mean stress p and a deviatoric stress:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix} + \begin{bmatrix} s_1 & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & s_2 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & s_3 \end{bmatrix}$$

where $s_1 + s_2 + s_3 = 0$. The second matrix on the right-hand side is the **stress deviator**. Its components are the deviatoric stresses. The **principal deviatoric stresses** are the amounts by which each of the principal stresses differs from the mean stress. They define the effective shear stress, which measures the intensity of the deviator:

$$\tau_{eff} = \left(\frac{1}{2} s_{ij} s_{ij}\right)^2 = \left[\frac{1}{2}\left(s_1^2 + s_2^2 + s_3^2\right) + \sigma_{23}^2 + \sigma_{31}^2 + \sigma_{12}^2\right]^{1/2}$$

The decomposition into the deviatoric stress s_{ij} and the volumetric stress $\delta_{ij}\bar{\sigma}$, utilizing the standard Kronecker delta, is written:

$$\sigma_{ij} = s_{ij} + \delta_{ij}\bar{\sigma}$$

and the normal stress relative to the mean stress is then described by the deviatoric stress:

$$s_{ij} = \sigma_{ij} - \delta_{ij}\bar{\sigma}$$

That is,

$$\delta_{11} = \delta_{22} = \delta_{33} = 1$$

$$\delta_{12} = \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0$$

under another form the matrix:

$$\begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}$$

is the identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In simpler words, where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ one can think of the rock being affected by two components:

- the mean stress $p = (\sigma_{11} + \sigma_{22} + \sigma_{33})/3$.
- and three deviatoric stresses : $s_1 = \sigma_{11} - p$

$$s_2 = \sigma_{22} - p$$

$$s_3 = \sigma_{33} - p$$

All shear stresses are deviatoric.
The main deviatoric stress s_1 is always positive, and the smallest one s_3 is negative (or equal to zero, with compression positive); the intermediate deviatoric stress is nearly equal to the mean stress. The positive deviatoric stress tends to shorten the rock in the direction of its action, while relative lengthening is easiest in the direction of the negative (tensional) deviator. Note that the deviatoric stress tensor always contains negative components.
As a corollary, only the deviatoric stresses leave permanent deformation in rocks.

Differential stress

The **differential stress** σ_d is the difference between the largest and the least principal stresses:

$$\sigma_d = (\sigma_1 - \sigma_3)$$

Its value, along with the characteristics of the deviatoric stress tensor, influences the amount and type of deformation a body experiences. Note that differential stress is a scalar. It should not be confused with the deviatoric stress, which is a tensor.

Stress acting on a given plane

In the following demonstration, it is essential to note that the value of stress varies with the orientation and magnitude of the imposed force, as well as the orientation and size of the area of action.
A force F acting on a real or imaginary plane P was resolved into components normal (F_N) and parallel (F_S) to the plane P. The components have magnitudes:

$$F_N = F\cos\theta \qquad \text{and} \qquad F_S = F\sin\theta \tag{2}$$

respectively.
We further consider that the cubic "point" previously used belongs to the plane P.
F is oriented to act normally to one cube face; for convenience, vertical F is on the top face of the cubic point. F is contained in the square, vertical section orthogonal to P through the cube. In this section, faces with unit area A are reduced to unit segment lengths.
By definition, stress is the concentration of force per unit area, which can be visualized as the intensity of force. The stress σ on the cube face has the magnitude:

Stress = Force / (Area of the cube face)

$$\sigma = F/A$$

The normal to plane P is inclined at an angle θ to F. The area A_P of the plane P is larger than the unit area A of the cube faces:

$$P_{(Area)} = \text{Cube face}_{(Area)} / \cos\theta$$
$$A_P = A/\cos\theta \qquad (3)$$

To know the magnitudes of the normal and shear components of stress across P, one must consider that A_P is the unit area. Hence, the normal components of force and stress acting on plane P are:

$$F_N = F\cos\theta = A\,\sigma\cos\theta = A_P\sigma\cos^2\theta$$

and the shear components: (4)

$$F_S = F\sin\theta = A\,\sigma\sin\theta = A_P\,\sigma\sin\theta\cos\theta$$

The general trigonometry states that:

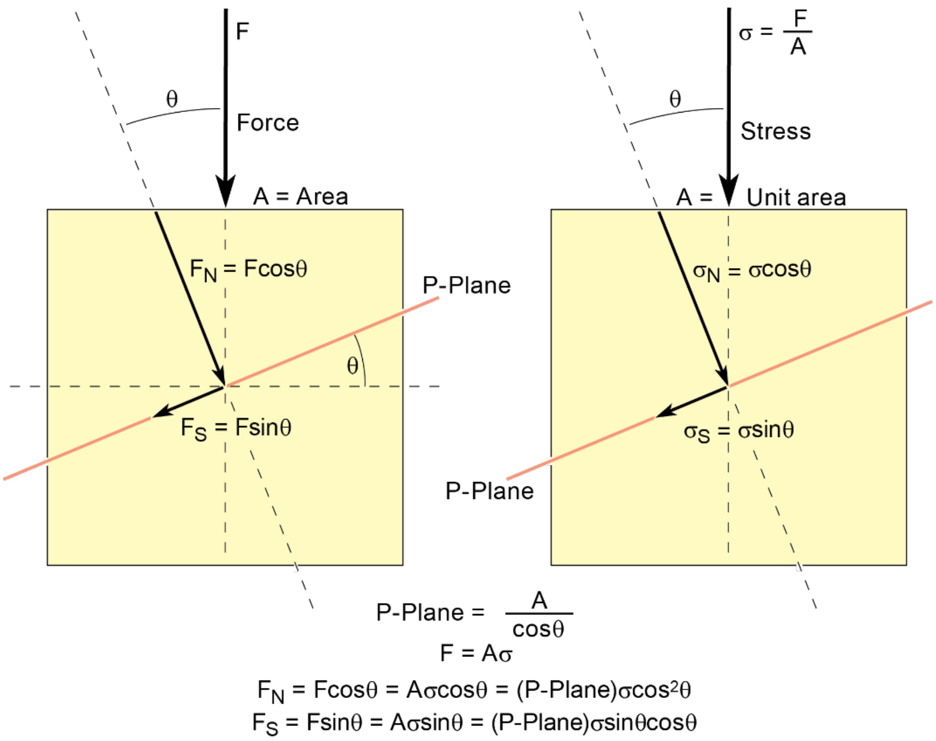
$$\sin\theta\cos\theta = (\sin 2\theta)/2$$

Equations (4) become:

$$\sigma_N = F_N/A_P = (F/A)\cos^2\theta = \sigma\cos^2\theta$$

and (5)

$$\sigma_S = F_S/A_P = (F/A)\sin\theta\cos\theta = (\sigma/2)\sin 2\theta$$



Comparison of Equations (2) and (5) shows that stresses cannot be resolved using vectors as if they were forces.
Typically, any rock is under a triaxial state of stress; σ_1, σ_2 and σ_3 are the principal stresses with $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

Remember! The convention in geology takes all positive stresses as compressive. In the non-geological literature stresses are considered positive in extension!

For practical purposes, one can take the arbitrary plane P parallel to σ_2, in turn parallel to the horizontal x-axis of the Cartesian coordinates. The angle θ between the line normal to P and the vertical σ_1 (parallel to the coordinate z-axis) is also the angle between the plane P and σ_3. One approaches the problem by considering a two-dimensional state of stress. For this simplification, one only considers the two-dimensional principal plane (σ_1,σ_3) while ignoring σ_2, orthogonal to this slicing plane. This simplification is consistent with the statement that it is the difference between σ_1 and σ_3 that rules deformation while σ_2 does little and can, as a first approximation, be discarded. One also considers that all lines in the (σ_1,σ_3) plane represent traces of planes perpendicular to it, thus parallel to σ_2.
From equations (5) that stress components due to σ_1 are:

$$\sigma_{1N} = \sigma_1 \cos^2\theta$$
$$\sigma_{1S} = \sigma_1 \sin\theta\cos\theta$$

σ_3 is orthogonal to σ_1. We can use the same trigonometric construction to resolve stress components due to σ_3 as:

$$\sigma_{3N} = \sigma_3 \sin^2\theta$$
$$\sigma_{3S} = \sigma_3 \sin\theta\cos\theta$$

Where the principal stresses are σ_1 and σ_3 the equations for the normal and shear stresses across a plane whose normal is inclined at θ to σ_1 are

$$\sigma_N = \sigma_1\cos^2\theta + \sigma_3\sin^2\theta$$
$$\sigma_S = \sin\theta\cos\theta(\sigma_1 - \sigma_3)$$

where the minus sign for σ_{3S} is necessary because the two stress directions point towards opposite directions along the plane.
From general trigonometry one knows the double angle identities:

$$\cos^2\theta = \frac{\cos 2\theta + 1}{2}$$
$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

which one can substitute in the previous equation to write the normal stress component:

$$\sigma_N = \sigma_1\left(\frac{\cos 2\theta + 1}{2}\right) + \sigma_3\left(\frac{1 - \cos 2\theta}{2}\right)$$

and simplify to:

$$\sigma_N = \frac{\sigma_1 + \sigma_3}{2} + \frac{\cos 2\theta(\sigma_1 - \sigma_3)}{2}$$

and one can write the shear stress component as:

$$\sigma_S = \frac{\sigma_1}{2}\sin 2\theta - \frac{\sigma_3}{2}\sin 2\theta$$

$$\sigma_S = \frac{1}{2}\sin 2\theta(\sigma_1 - \sigma_3)$$

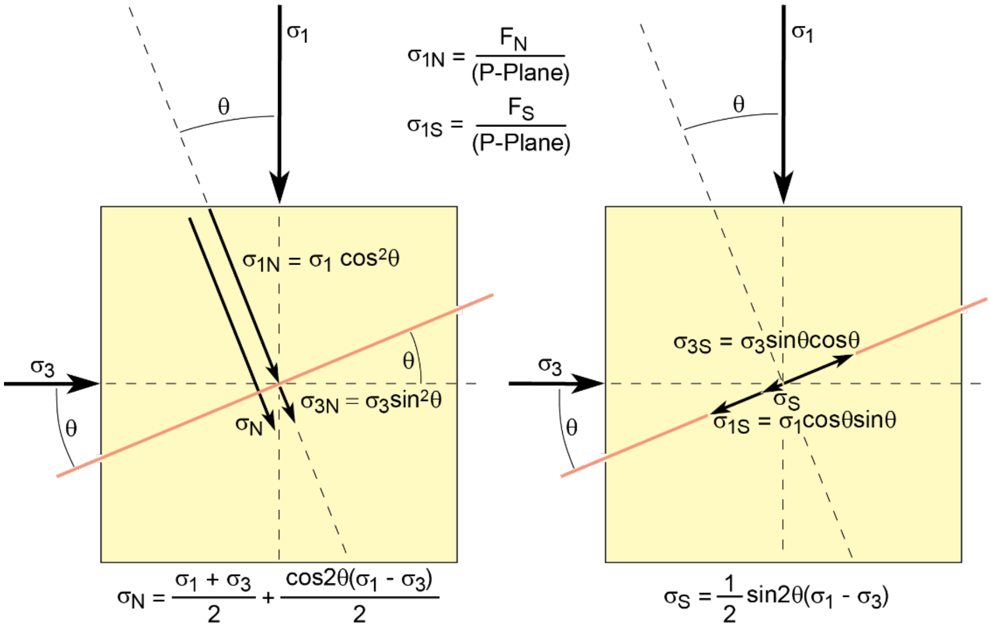
When the principal stresses are σ_1 and σ_3 the equations for the normal and shear stresses across a plane whose normal is inclined at θ to σ_1 are

$$\sigma_N = \frac{(\sigma_1 + \sigma_3)}{2} + \frac{(\sigma_1 - \sigma_3)\cos 2\theta}{2}$$

(6)

$$\sigma_S = \frac{(\sigma_1 - \sigma_3)\sin 2\theta}{2}$$

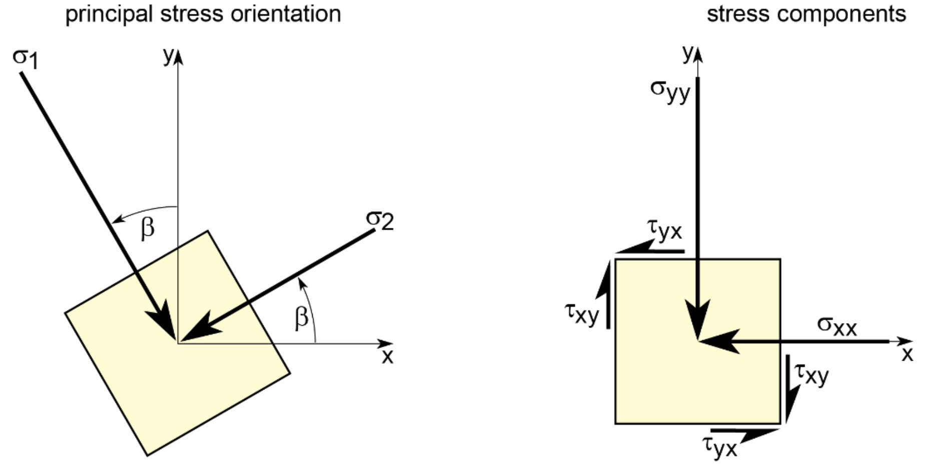
respectively.
Note that (6) reduces to (5) when σ_3 is zero. These relations are extensively used in geological studies because σ_1 and σ_3 are often close to the horizontal and vertical (lithostatic) tectonic stresses.



These equations demonstrate that the value of σ_S in (6) is maximum when $\sin 2\theta = 1$ i.e. $2\theta = 90°$. Thus, the planes of **maximum shear stress** make an angle of 45° with σ_1 and σ_3.
In all cases where $\sigma_1 \geq \sigma_2 \geq \sigma_3$ the planes of maximum shear stress are only two in number and intersect along σ_2. Indeed, it has been observed in triaxial tests (σ_1, σ_2 and σ_3 have non zero magnitudes) that shear fractures form angles close to 45° to the principal stress axis σ_1. The paired faults, called **conjugate faults,** develop more or less synchronously in both of the equally favored orientations. Remember that conjugate faults intersect in a line parallel to the intermediate principal stress axis σ_2. Normal compressive stresses on these planes tend to inhibit sliding along this plane; shear stresses on these planes tend to promote sliding.
In the special situation where $\sigma_2 = \sigma_3$ or $\sigma_1 = \sigma_2$ there is an infinite number of such planes inclined at 45° to σ_1 or σ_3, respectively.
In all cases, the maximum shear stress has the value $(\sigma_1 - \sigma_3)/2$.
Equation (6) also implies that for any arbitrary state of stress with σ_1 and σ_3, there are surfaces on which no shear forces are exerted. We will use this property to define directions of principal stresses.

Relationship between normal stress and shear stress: Mohr circle

The principal stresses are those that are orthogonal to the three mutually orthogonal planes on which shear stresses vanish to zero. Between these special orientations, the normal and shear stresses vary smoothly with respect to the rotation angle θ. How are the normal and shear stress components associated with direction?



Stresses on a square point in the two-dimensional Cartesian coordinate system

Analytical demonstration

Rearranging σ_N of equations (6) and squaring both equations one gets:

$$\left[\sigma_N - (1/2)(\sigma_1 + \sigma_3)\right]^2 = \left[(1/2)(\sigma_1 - \sigma_3)\right]^2 \cos^2 2\theta$$

$$\sigma_S^2 = \left[(1/2)(\sigma_1 - \sigma_3)\right]^2 \sin^2 2\theta \tag{7}$$

One can add both equations (7) to write

$$\left[\sigma_N - (1/2)(\sigma_1 + \sigma_3)\right]^2 + \sigma_S^2 = \left[(1/2)(\sigma_1 - \sigma_3)\right]^2 \left(\cos^2 2\theta + \sin^2 2\theta\right)$$

Since $\cos^2 + \sin^2 = 1$ for any angle:

$$\left[\sigma_N - (1/2)(\sigma_1 + \sigma_3)\right]^2 + \sigma_S^2 = \left[(1/2)(\sigma_1 - \sigma_3)\right]^2$$

in which one recognizes the form of the standard equation of a circle in the coordinate plane (x,y) with centre at (h,k) and radius r :

$$(x - h)^2 + (y - k)^2 = r^2$$

with $x = \sigma_N$, $y = \sigma_S$ and $k = 0$.

This leads to the two-dimensional representation of stress equations known as the **Mohr diagram**.

The radius of the stress-circle is: $\left(\dfrac{\sigma_1 - \sigma_3}{2}\right)$

The center of the stress-circle on the σ_N axis is at h : $\left(\dfrac{\sigma_1 + \sigma_3}{2}\right)$

Cyclic interchange of the subscripts generates two other circles for the other two principal stress differences, $(\sigma_2 - \sigma_3)$ and $(\sigma_1 - \sigma_2)$.

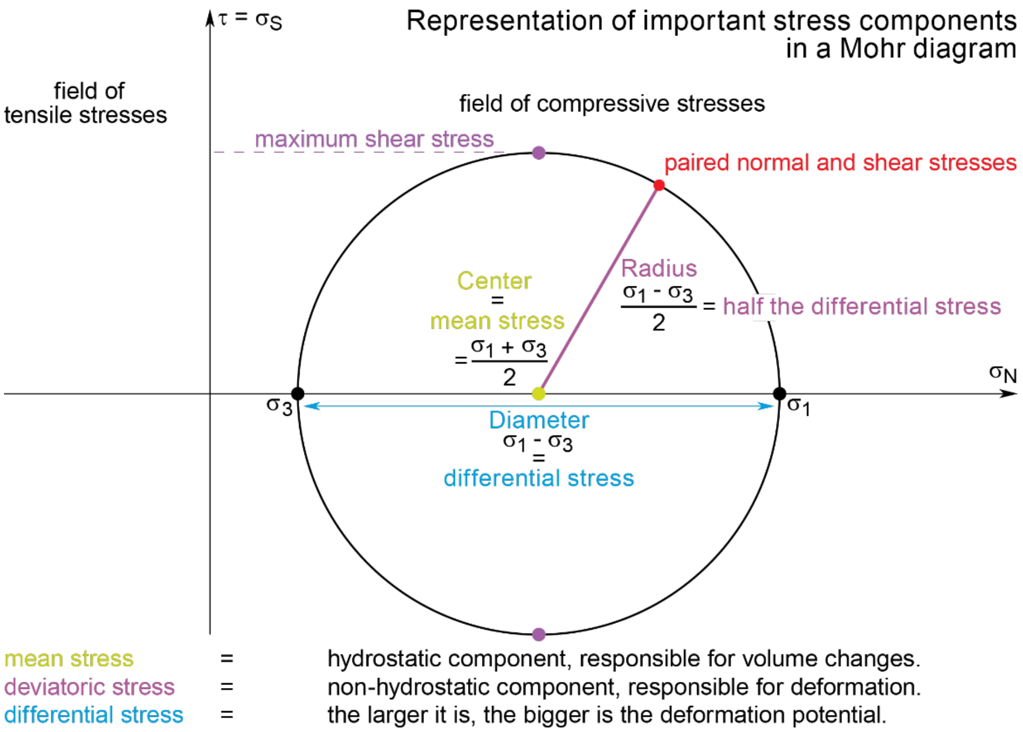
Graphical construction

Equations (6) describe a circular locus of paired values σ_N and σ_S (the normal and shear stresses, respectively) that operate on planes of any orientation within a body subjected to known values of σ_1 and σ_3. In other word, for a given state of stress and orientation, normal and shear stresses on a plane are represented by a point on the Mohr circle.

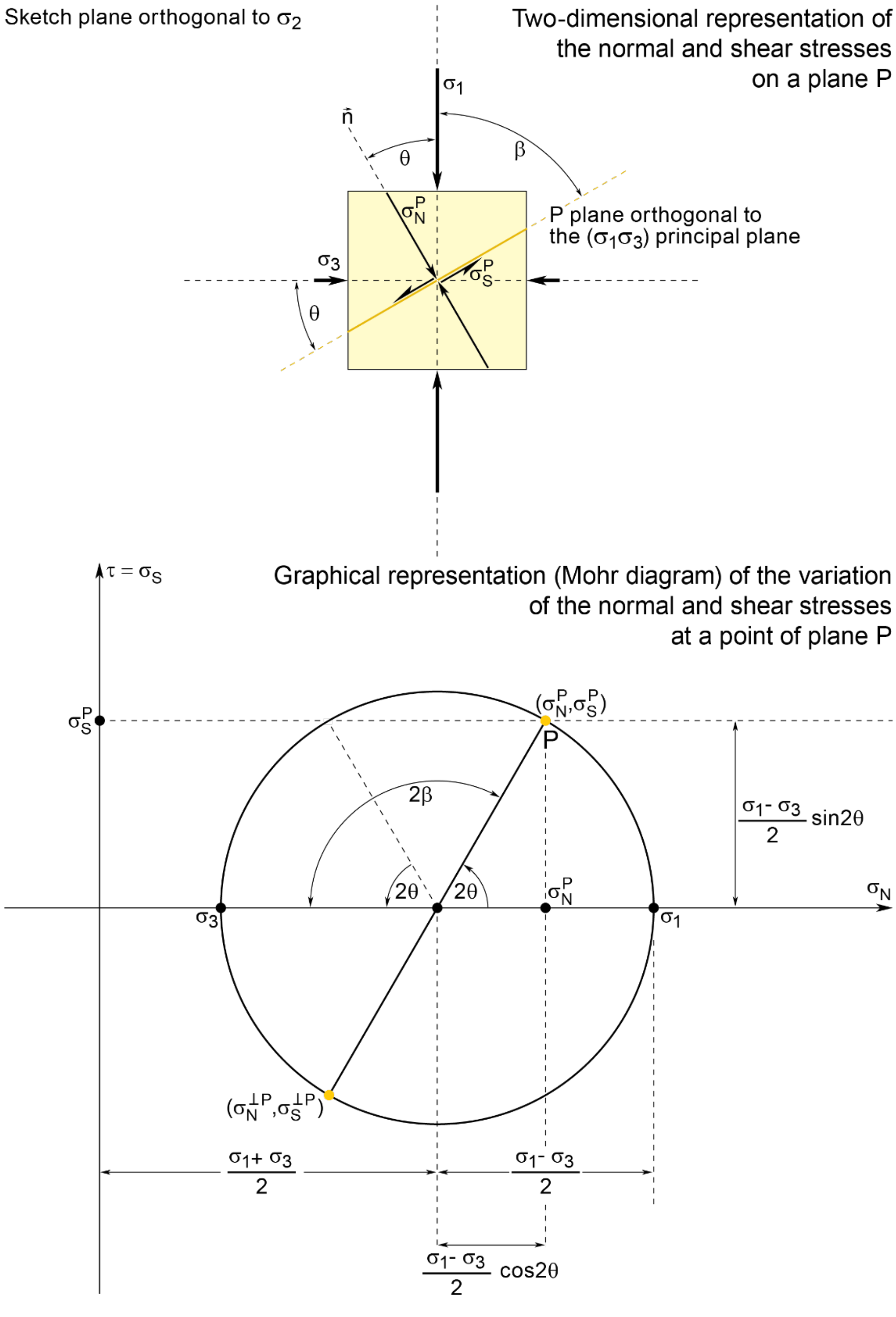
Stress in two dimensions (plane stress)

The construction of the Mohr stress circle proceeds as follows:

- For the two-dimensional stress, the normal and shear stresses can be plotted along two orthogonal, scaled coordinate axes, with normal stresses σ_N along the abscissa (or horizontal x-axis), and shear stresses σ_S along the ordinates (or vertical y-axis).
- These axes have no geographic orientation but have positive and negative directions.
- By convention, the right half of the diagram is positive for compressive normal stresses. Shear stresses that have an anticlockwise sense (consistent with the trigonometric sense) are considered positive and are plotted above the abscissa axis.



- The principal stresses σ_1 and σ_3 of a given ~~a~~ state of stress are per definition normal stress components; they are both plotted along the abscissa, at a distance from the origin equal to their numerical values.
- A circle with the diameter $(\sigma_1 - \sigma_3)$ and center at $C = [(\sigma_1 + \sigma_3)/2]$ is constructed through points σ_1 and σ_3. The maximum principal stress σ_1 is at the right extremity of the circle, the least principal stress value σ_3 is the left extremity of the circle.



The circumference of the circle is the locus of all possible paired values of σ_N and σ_S. Therefore, any point P on the circle has coordinates (σ_N, σ_S) where σ_N and σ_S are given by equations (6); 2θ is the angle between the σ_N axis and the line PC measured in the anticlockwise (trigonometric) sense from the right-hand end of the σ_N axis. The coordinates of any point P on the circle give the normal stress σ_N (read along the abscissa) and shear stress σ_S (read along the ordinates) across a plane whose normal (ATTENTION: not the plane itself) is inclined at θ to σ_1. For simple geometrical reasons, θ is also the angle between the fault plane and the least stress σ_3. The 2β angle measured clockwise from σ_3 to the PC radius is twice the angle between σ_1 and the actual fault plane.

This construction may also be used to find σ_1, σ_3, and θ given σ_N and σ_S on two orthogonal planes. Since angles are doubled in this graphical format, the point representing the plane orthogonal to P is opposed to it. The Mohr circle can be constructed through these two points linked by a diameter intercepting the horizontal axis at center C.

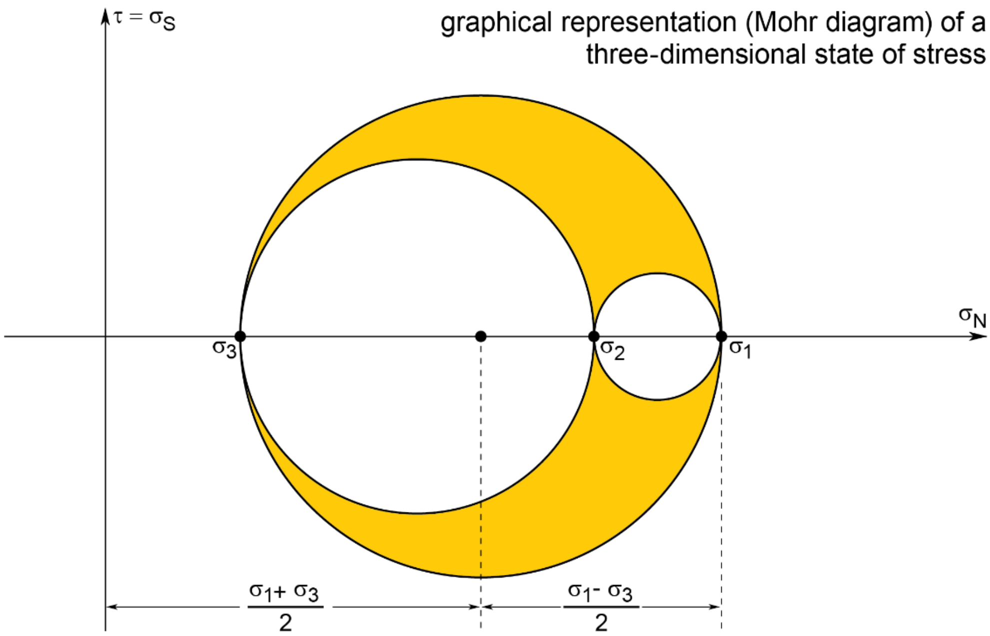
The Mohr diagram thus allows the magnitudes of normal and shear stresses on variously oriented planes to be plotted together. It neatly shows that:

- The points on the circle (hence attitude of planes) along which shear stress σ_S is greatest correspond to values of $\theta = \pm 45°$.
- The maximum stress difference $(\sigma_1 - \sigma_3)$ determines the value of the greatest σ_S simply because the differential stress is the diameter of the Mohr circle, which is twice the vertical radius, the maximum shear stress, the radius keeps the same magnitude regardless of the coordinate orientation. It is the second invariant of the two-dimensional stress tensor.

Stress in three dimensions

The Mohr construction applied to three-dimensional stress states has three circles: two small circles (σ_1, σ_2) and (σ_2, σ_3) are tangent at σ_2 and lay within the larger (σ_1, σ_3) circle.

The three diagonal components σ_1, σ_2 and σ_3 of the stress tensor are normal stresses plotted along the horizontal axis; non-diagonal components are shear stresses plotted along the vertical axis. All possible (σ_N, σ_S) points plot on the large (σ_1, σ_3) Mohr circle or between this circle and the (σ_1, σ_2) and (σ_2, σ_3) Mohr circles. The diagram area between these three circles is the locus of stress on planes of all orientations in three dimensions.



Mohr diagrams are used extensively in discussions of the fracturing of rock masses because they graphically represent the variation of stress with direction and allow finding the stresses on known weak planes.

Exercise:
Draw Mohr images of the following state of stress:
Hydrostatic, uniaxial, axial, triaxial

Effects of pore fluid pressure

Fluid pressure

Fluid pressure refers to the pressure in and exerted by fluids contained in the cracks and pores of granular materials. Rocks within depths of a few kilometers of the crust commonly have either intergranular or fracture porosity along which a column of fluids exists up to the surface. If the fluid reservoir is in static equilibrium, the fluid pressure P_f is closely approximated by the equation:

$$P_f = \rho_{(f)} gz$$

where $\rho_{(f)}$ is the fluid density, g the acceleration of gravity, and z the depth. This is the **hydrostatic pressure**, which differs from the lithostatic pressure (weight of rocks at the same depth).

Exercise:
Calculate the gradient of hydrostatic pressure due to a column of pure water and that of the lithostatic pressure.

10 MPa/km – 23-27 MPa/km

In practice, measured fluid pressures are occasionally less than but more often greater than the normal hydrostatic pressure. Overpressured fluids are attributed to one or more of several mechanisms such as compaction of sediments, diagenetic/metamorphic dehydration of minerals, artesian circulation. Tectonic stresses in active areas may also increase the interstitial water pressure. Carbon dioxide released from the mantle or other sources is a common abnormally pressured fluid.

Effective stress

The **total stress field** in a porous solid can be specified in terms of normal and shear components across plane surfaces. The solid and its interstitial fluid combined exert the total normal stresses and the total shear components. The tensor of total stress (equation 1) expresses the total stress field:

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

in which the stresses due to the fluids have also nine components:

$$\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

in which:

$$p_{ii} = p_{jj} = p$$
$$p_{ij} = p_{ji} = 0$$

because normal (hydrostatic) pressures are equal in all directions and shear stresses of the pore fluids are neglected since they are much smaller than those in the solid. Therefore, a diagonal, isotropic matrix (a square matrix that has non-zero elements only along the main diagonal) represents the fluid pressure:

$$\begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}$$

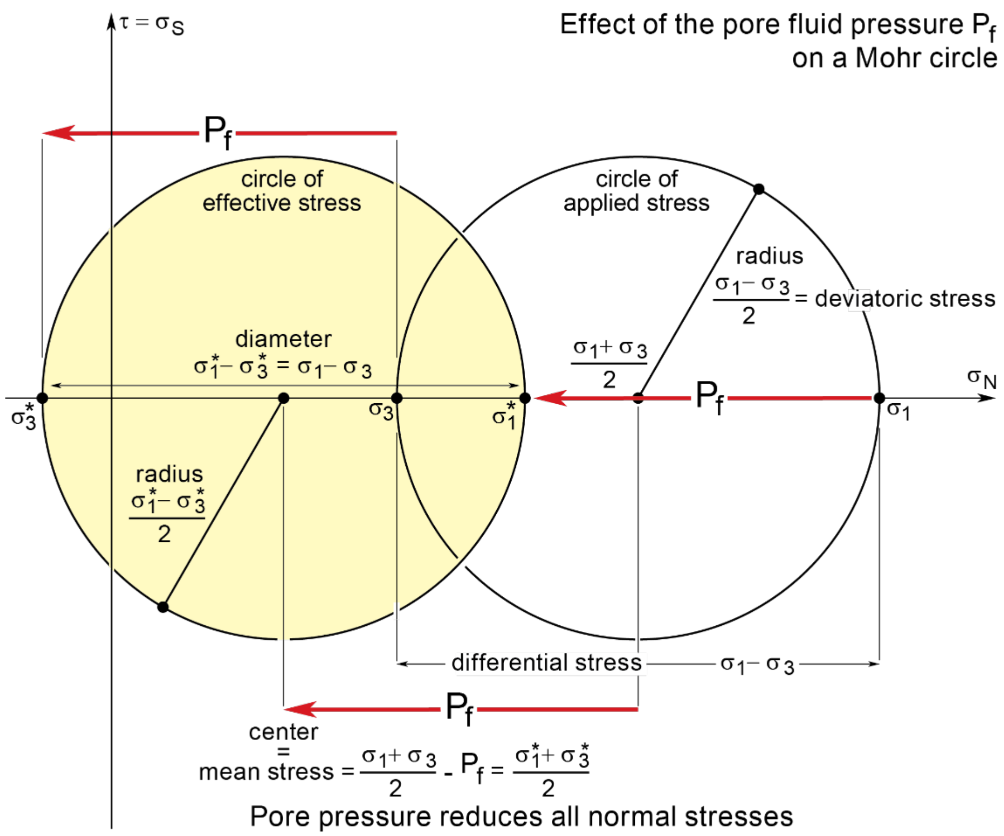
The **effective stress** is the difference between the total stress and fluid pressure $P_f = p$:

$$\sigma_{eff} = \sigma_{tot} - P_f$$

The principal effective stresses σ_1^{eff}, σ_2^{eff} and σ_3^{eff} concern the solid part of the porous medium only.

Changes in confining pressure

If a material contains a fluid under pressure P_f, this pressure counteracts with equal intensity in all directions the principal stresses due to an applied load. Therefore, the values of all normal stresses on any plane are reduced by the value of the fluid pressure P_f. The values of all shear stresses remain the same, indicating that they are independent of the hydrostatic component. In rocks, it corresponds to a change in confining pressure.



The **effective mean stress** is the difference between the mean stress and the fluid pressure:

$$\sigma_{eff} = \frac{\sigma_1^{eff} + \sigma_2^{eff} + \sigma_3^{eff}}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} - P_f = \bar{\sigma} - p$$

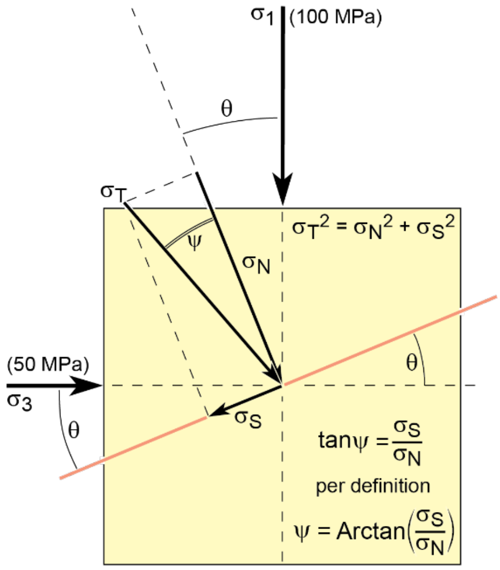
In a Mohr circle representation, changes in the total normal tress $(\sigma_1 + \sigma_3)$ shift the circle along the abscissa axis by an amount equal to the pore fluid pressure P_f, without changing its diameter, i.e. the circle moves to the left, towards smaller values of normal stresses and keeps its size. The circle would move to the right for a decrease in pore pressure.

Stress Ellipsoid

The stress ellipsoid is a graphical representation of the six parameters of the stress tensor. Three mutually perpendicular directions, called the **principal directions**, and the intensities of the stresses in these directions are called the **principal stresses**.

Numerical Approach in two dimensions

The stress system is first restricted to a two-dimensional plane.



Exercise (to be done with Excel or Matlab)

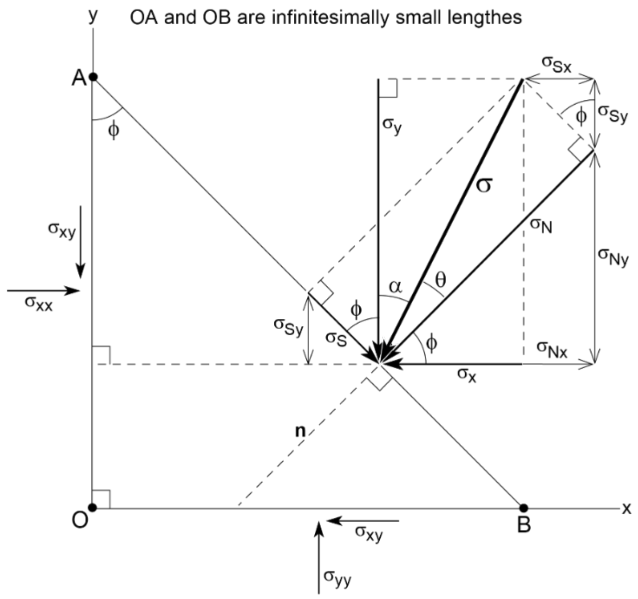
** Take a point O on a horizontal plane P.*

A vertical stress of 100 MPa and a horizontal stress of 50 MPa act at O.

** Using a calculator and the figure below, determine the absolute values of stresses for planes inclined at 5° intervals through O.*

** Make separate calculations for the normal, shear and total stresses.*

** Describe variation of stress magnitudes as a function of orientations.*



Geometrical relationship between the components of stress vectors and the components of stress tensors

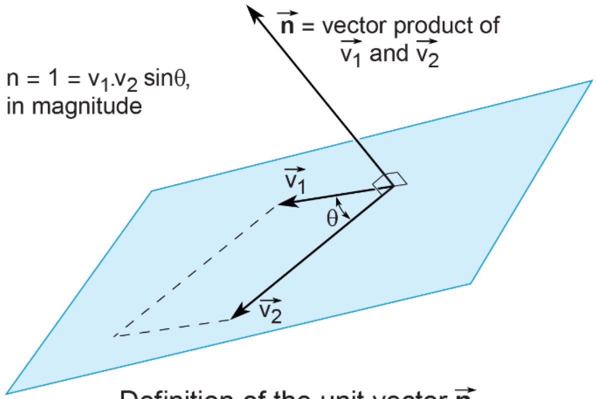
An ellipse is generated for the total stress, known as the stress ellipse, which is a reduction of the stress ellipsoid.
We can imagine the same exercise

- on a vertical plane where the vertical stress is 100 MPa and the horizontal stress has an intermediate value of 75 MPa.
- On a horizontal plane where the intensity of the two perpendicular stresses are 75 and 50 MPa, respectively.

The combination of these three stress ellipses around O generates the stress ellipsoid.

Analytical approach

We now consider stress σ acting across a plane P, within the rectangular Cartesian coordinate directions Ox, Oy and Oz, with Oz vertical.



Definition of the unit vector $\vec{n}$, perpendicular to the plane containing vectors $\vec{v_1}$ and $\vec{v_2}$

Let **n** be the unit normal vector defining the plane P and piercing the plane through the point P. The direction of the line OP, thus the direction of the plane P, can be expressed by the spherical coordinates of **n**, which are:

$$\begin{cases} n_x = \sin\theta . \cos\phi \\ n_y = \sin\theta . \sin\phi \\ n_z = \cos\theta \end{cases} \tag{8}$$

The spherical directions are equal to the direction of cosines $\{\cos\alpha ; \cos\beta ; \cos\theta\}$, directly obtained from the angles between the line normal to the plane and the coordinate axes. Since **n** is a unit vector, these components must satisfy the unit length condition:

$$n_x^2 + n_y^2 + n_z^2 = 1 \tag{9}$$

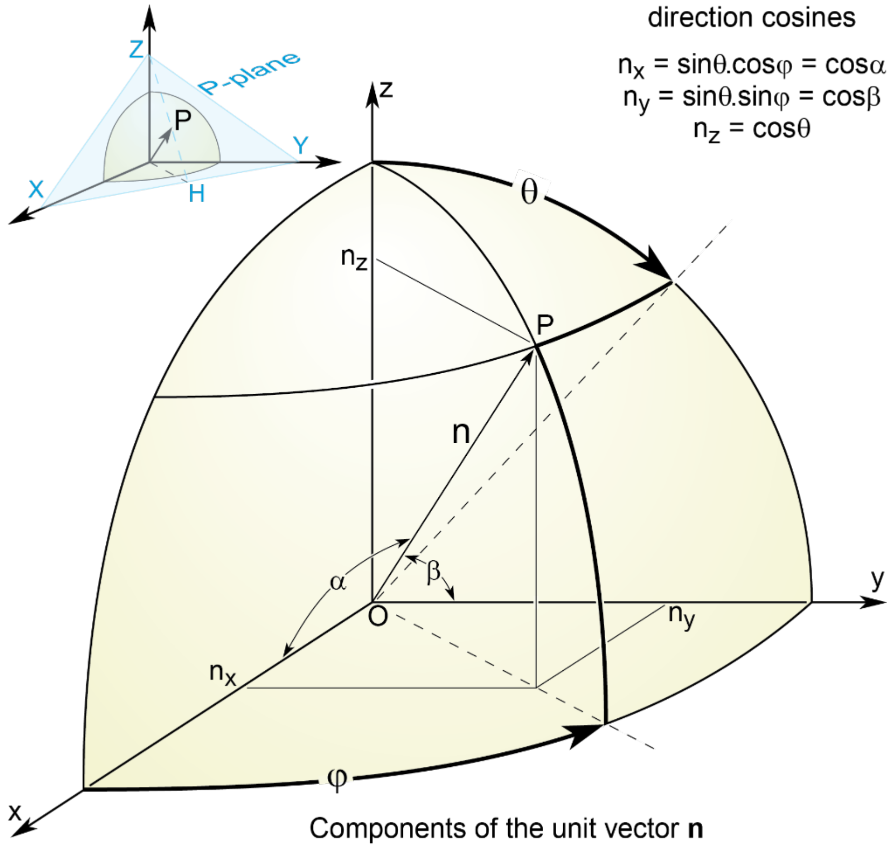
Now one considers the infinitely small tetrahedron bounded by the plane P and by the three other triangular faces containing the coordinate axes. The plane cuts the axes Ox, Oy and Oz at points, X, Y and Z, respectively. The triangle XOY is the projection parallel to Oz of the face XYZ onto the plane xOy. The area of the XYZ face is:

(1/2).(base XY * height ZH)

The area of the XOY face is:

(1/2).(base XY * height OH)

The ratio between these two faces is simply the ratio OH/ZH, the two sides of the same triangle with a right angle in O. A geometrical construction in the plane ZOH shows that OH/ZH = $\cos\theta = n_z$. The similar reasoning for the projection of XYZ on to the other two coordinate planes shows that the proportionality factors are α (projection parallel to Ox) and β (projection parallel to Oy).



Components of the unit vector **n**
orthogonal to the P-plane tangent to the unit sphere at point P

We now consider equilibrium, which means the balance of forces acting on the infinitely small tetrahedron under consideration. Forces applied to each face are decomposed into one normal and two shear forces.

$$\begin{matrix} \Phi_{xx} & \Phi_{xy} & \Phi_{xz} \\ \Phi_{yy} & \Phi_{yx} & \Phi_{yz} \\ \Phi_{zz} & \Phi_{zx} & \Phi_{zy} \end{matrix}$$

We take XYZ as the unit area on which the force applied F/1 is also a stress vector $\mathbf{T}$ whose components parallel to the coordinate axes are T_x, T_y and T_z. $\mathbf{T}$ is then determined by the simple vectorial sum where the cosine directions weigh the vector components:

$$\mathbf{T} = T_x \cos\alpha + T_y \cos\beta + T_z \cos\theta \tag{10}$$

These three components are each balanced by force components acting in the same direction on the three other faces. For example:

$$(\text{area} = 1)T_x = \Phi_{xx} + \Phi_{yx} + \Phi_{zx} \tag{11}$$

Areas of the coordinate faces with respect to the XYZ unit area were calculated above as $\cos\alpha$, $\cos\beta$ and $\cos\theta$. Per definition, force components ϕ_{ij} are stress components σ_{ij} and τ_{ij} multiplied by the areas on which they are applied. Then one can write all force components as:

$$\begin{matrix} \Phi_{xx} = n_x\sigma_{xx} & \Phi_{xy} = n_x\tau_{xy} & \Phi_{xz} = n_x\tau_{xz} \\ \Phi_{yy} = n_y\sigma_{yy} & \Phi_{yx} = n_y\tau_{yx} & \Phi_{yz} = n_y\tau_{yz} \\ \Phi_{zz} = n_z\sigma_{zz} & \Phi_{zx} = n_z\tau_{zx} & \Phi_{zy} = n_z\tau_{zy} \end{matrix}$$

Equation (10) becomes:

$$T_x = n_x\sigma_{xx} + n_y\tau_{yx} + n_z\tau_{zx} \tag{12}$$

Absence of rotation implies that $\tau_{ij} = \tau_{ji}$ and equation (11) becomes:

$$T_x = n_x\sigma_{xx} + n_y\tau_{xy} + n_z\tau_{xz} \tag{13}$$

With similar equilibrium arguments along the other directions, the coordinate axes of the stress vector T relative to **n** are:

$$\begin{cases} T_x = \sigma_{xx}n_x + \tau_{xy}n_y + \tau_{xz}n_z \\ T_y = \tau_{yz}n_x + \sigma_{yy}n_y + \tau_{yz}n_z \\ T_z = \tau_{zx}n_x + \tau_{zy}n_y + \sigma_{zz}n_z \end{cases}$$

These 3 linear equations are written in matrix notation:

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} n_x \\ n_y \\ n_z \end{bmatrix} \tag{14}$$

The 3 X 3 stress matrix that transforms linearly every column vector as **n** into another column vector as **T** defines the second-order **stress tensor**. It links any given plane with its associated stress vector. It is written in a condensed fashion as the Cauchy's formula:

$$\sigma_i = \sigma_{ij}n_j$$

The magic of this formula: Multiplying the stress tensor, treated as a simple matrix, by a unit vector n_j, which is the normal to a certain plane, one gets the traction vector acting on that plane.

Remember: If **A** and **B** are two rectangular arrays of variables, their product **C** is defined as:

$$\mathbf{C} = \mathbf{A.B}$$

where the elements c_{ij} of **C** is obtained from the i^th^ row of **A** and the j^th^ column of **B** by multiplying one by the other, element by element, and summing the product:

$$c_{ij} = \sum_k a_{ik}b_{kj}$$

AB is defined only if the width (number of columns) of **A** is equal to the height (number of rows) of **B** and, in general, **AB** ≠**BA**. Finally, **AB**=0 does not imply that either **A** or **B** is a null matrix.

The parallelogram construction and equation (12) show that the normal stress σ across the plane with the normal **n** is given by:

$$\mathbf{T.n} = T_x n_x + T_y n_y + T_z n_z$$

$$\sigma = n_x{}^2\sigma_x + n_y{}^2\sigma_y + n_z{}^2\sigma_z + 2\left(n_y n_z\tau_{yz} + n_x n_z\tau_{zx} + n_x n_z\tau_{xy}\right) \tag{15}$$

while the corresponding shear stress is:

$$\tau^2 = \mathbf{T}^2 - \sigma^2$$

We are looking for a geometric representation of the variation of stress with direction. The theory can be followed most easily for a two-dimensional stress system where the angle xOP = θ. Then $n_x = \cos\theta$, $n_y = \sin\theta$ and $n_z = 0$. Equation (9) reduces to:

$$\sigma = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

In the vertical plane xOz, taking σ parallel to the horizontal axis Ox, the only forces acting on the tetrahedron (prism in two-dimension) in the Ox direction are σ_x * (area of *P*) and σ_1 * (area of xOz). These forces must balance for the equilibrium of the tetrahedron, so that

$$\sigma_x = \sigma_1 \text{ (area of xOz / area of } P\text{)}$$

Now, since:

$$\text{Area of xOz} = n_x * \text{(area of } P\text{)},$$

(areas are reduced to lines in this 2D projection) one gets:

$$\sigma_x = n_x \sigma_1$$

and, by similar arguments,

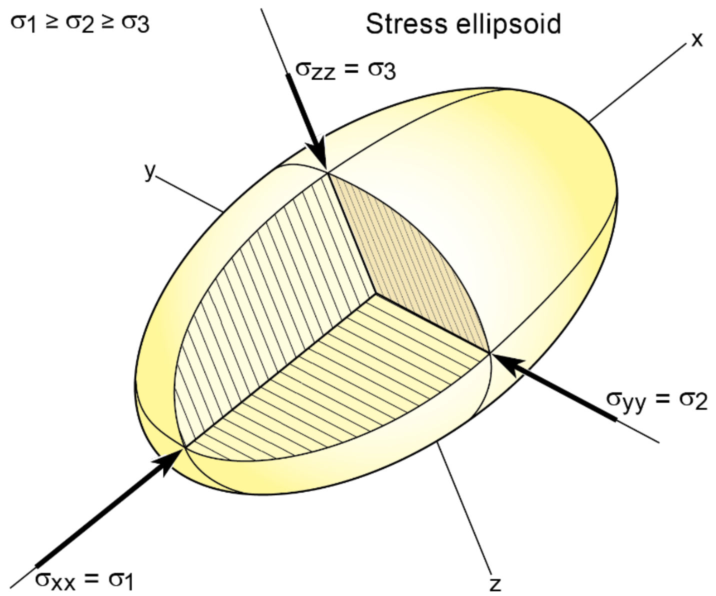
$$\sigma_y = n_y \sigma_2$$

$$\sigma_z = n_z \sigma_3$$

Substituting cosine directions from equation (9), it follows that:

$$\left(\sigma_x^2 / \sigma_1^2\right) + \left(\sigma_y^2 / \sigma_2^2\right) + \left(\sigma_z^2 / \sigma_3^2\right) = 1 \tag{16}$$

Equation (16) is the equation of an ellipsoid centered at the origin with its axes parallel to the coordinate axes. The ellipsoid semiaxes are in the same direction and have the same magnitudes as the principal stresses.



This **stress ellipsoid** is a commonly used graphical representation of stress. Its principal axes are known as the **principal axes of stress**, which are mutually perpendicular directions of zero shear stress. The direction and magnitude of a radius vector of the stress ellipsoid give a complete representation of the stress across the plane conjugated to that radius vector. The radius vector is:

$$s = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}$$

The ellipsoid is the loci of all s-extremities. Notice that, in general, the plane corresponding to a given radius vector is not normal to the radius vector.
Intuitively, from the known symmetries of the ellipsoid, there are always 3 orthogonal directions (the principal axes) for which **T** and **n** have the same direction. The normal stress of equation (14) across a plane whose normal has direction cosines $\{n_x; n_y; n_z\}$ is now given by:

$$\sigma = n_x{}^2\sigma_1 + n_y{}^2\sigma_2 + n_z{}^2\sigma_3$$

The magnitude of the shear stress across this plane is:

$$\tau^2 = (\sigma_1 - \sigma_2)^2 n_x{}^2 n_y{}^2 + (\sigma_2 - \sigma_3)^2 n_y{}^2 n_z{}^2 + (\sigma_3 - \sigma_1)^2 n_z{}^2 n_x{}^2$$

If the directions are taken as coordinate axes, all shear components are null. The stress tensor (equation 14) is then simplified to:

$$T = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

Stress ellipsoids –stress tensor

The numerically generated ellipse represents a section of the **stress tensor** in an **ellipsoidal** form within a specific principal plane. The stress tensor is not a single vector. It refers to the entire collection of stresses acting on every plane of every conceivable orientation passing through a discrete point in a body, specifically the center of the ellipsoid, at a given instant in time.

To describe the stress tensor, one needs the orientation, size, and shape of the stress ellipsoid. Thus, one needs to determine the orientations and lengths of the three principal axes of the stress ellipsoid. If one can define the stress tensors at each and every point within a body, one can fully describe a stress field, which is the entire collection of stress tensors. This exercise is a fundamental approach for evaluating the relationship between stress and strain.

Stress field

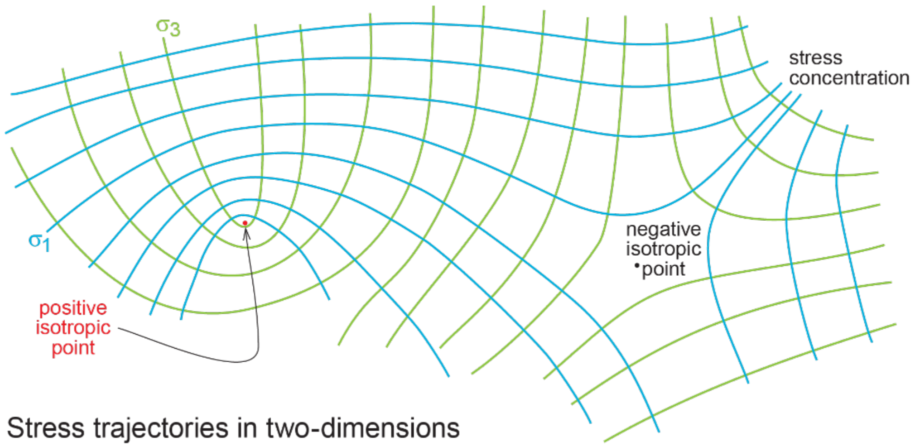
When surface forces are applied to a body, the resulting stresses within this body generally vary in direction and intensity from point to point. The **stress field** refers to the distribution of stresses at all points throughout the body. The stress field can be portrayed either as a set of stress ellipsoids, or as their stress axes, or as stress trajectories. The stress field is **homogeneous** if both the normal and shear components are the same in magnitude and orientation at all points. Otherwise, and commonly in geology, it is **heterogeneous**. The relative uniformity of stress orientation and magnitudes is striking, allowing for the mapping of regional stress fields.

See the **World Stress Map**: http://dc-app3-14.gfz-potsdam.de/

Two or more stress fields of different origins may be superimposed to give a **combined stress field**. The sources of the stress are manifold, and consequently, stress is unevenly distributed within the Earth's lithosphere. Its magnitudes are highest within, or next to, the regions where causative forces are exerted. Stress gradually diminishes away due to the elastic and creeping strain energy consumed in deforming rocks. The **stress gradient** is the rate at which stress increases in a particular direction, for instance, depth with a normal hydrostatic gradient = 10 MPa/km and an overburden gradient = 23 MPa/km. Curves of iso-stress-magnitude (stress contours) illustrate such gradients. In the lithosphere, stresses result from forces that are transmitted from point to point. Knowing the magnitude and the orientation of principal stresses at any point allows for calculating the normal and shear components on any plane passing through this point.

Stress trajectories

In two dimensions, on a given surface (e.g., in map view), the stress trajectories are virtual curves that image the directions of the principal stresses at all points and link the stress axes of the same class. For example, one set of lines determines the direction of the maximum principal stress, and a second set determines that of the minimum principal stress. These two sets are everywhere orthogonal. Individual trajectories may be curved, but principal stresses must remain at right angles to each other at every point along the curve. Then, stress trajectories portray continuous variation in principal stress orientation from one point to another through the body.



Stress trajectories in two-dimensions

Adjacent trajectories coming closer together indicate stress concentration.
Principal stresses are equal at **isotropic points**. Entwining stress trajectories bound positive isotropic points; dissociating trajectories define negative isotropic points.
Principal stresses are all zero at **singular points**.

Slip lines

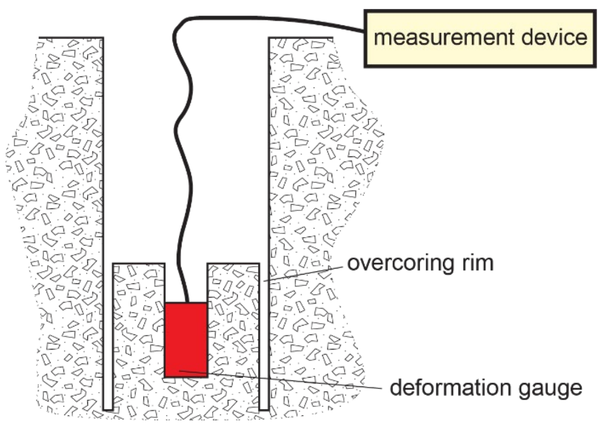
Knowing the principal stress trajectories, potential shear surfaces at any point in the stress field are the surfaces tangent to the direction of maximum shearing stress at that point. The traces of these potential shear surfaces are called **slip lines.** In two dimensions, two sets of lines represent curves of dextral and sinistral senses of shear. They converge towards isotropic points.

Stress Measurement

The motivation to measure stresses stems from geological hazards, engineering activity, and resource exploration. The stresses initially supported by the rock excavated from mines and boreholes are immediately transferred to the surrounding rocks. The resultant stress concentration is well understood from elastic theory, so stress measurement can be made indirectly by measuring the rock response around the borehole or the mine. The two primary methods for measuring in situ stresses using stress concentration around boreholes are near-surface **overcoring** and **hydraulic fracturing**.

Elastic strain: Overcoring and breakout

Overcoring consists in installing a strain gauge on the bottom of a tubular borehole. Then a coaxial and annular hole is drilled around and deeper with an internal radius smaller than the first hole.



Principle of stress measurement by overcoring

This procedure releases stresses from the rock cylinder bearing the gauge, which has become isolated from the regionally stressed surrounding. The subsequent elastic deformation of the circular inner core into an elliptical one defines the orientation of horizontal stresses, with the long axis of the ellipse parallel to the maximum horizontal principal stress, resulting in more relaxation in the most compressed direction. This strain can be converted into a stress magnitude if the elastic properties of the rock are known.
After drilling, a circular borehole may become elliptical, or breakout, in response to stresses in the surrounding rock. The long axis of the ellipse is parallel to the minimum horizontal stress.

Hydraulic fracturing

Hydraulic fracturing involves injecting fluids into a sealed well and pressurizing it until a fracture is generated in the surrounding rock, which is then filled with the coring fluid. The fluid pressure required to induce tensile fracture at the wall of the borehole is the **breakdown pressure**. Fracturing will occur if it is equal to the tensile strength σ_T of the rock, which in turn is assumed to be equal to the magnitude of the minimum horizontal effective stress $\sigma^*_{h,eff}$:

$$\sigma^*_{h,eff} = -\sigma_T$$

The concept assumes that one principal stress is vertical (representing a near-surface condition) and aligned with the vertical wellbore. The magnitude of the vertical stress σ_v is the weight of overlying rocks. The goal is to find the magnitudes of the greater (σ_H) and smaller (σ_h) principal stresses in the horizontal plane and their orientation. Hydraulic fracturing assumes that cracks form perpendicular to the minimum horizontal stress. Hence, measuring the orientation of created hydraulic fractures and the breakdown pressure provides insight into the stress tensor.
However, this technique does not specify the direction of principal stresses. Hydraulic fracture initiation also depends on stress regimes and wellbore orientation.
Once the injection is ceased, the propped fracture becomes a passage for hydrocarbon gas or water flow from the drilled reservoir to the well, thus allowing enhanced production. Hydraulic fracturing is a practical stimulation technique used to enhance hydrocarbon recovery from low-permeability reservoirs.

Focal mechanisms

If an earthquake corresponds to slip on a fault with near optimal orientation for reactivation with respect to the regional stress field, the P (for com**P**ression, the ground moves towards the seismic station), B and T (dila**T**ation, motion away from the station) axes define the elastic strain released in the earthquake. Their orientation approximates the active principal stress directions s_1, s_2 and s_3, respectively.

Present-day tectonic stress field

Stress determination is both incomplete and sparse. Results show that the stress state is characteristically heterogeneous and unpredictable in space and time. Extrapolations from individual measurements remain very limited in simplifying the real information because local geological features govern local perturbations. However, syntheses demonstrate the existence of remarkably uniform "**Andersonian**" stress provinces, with two of the principal stresses, horizontal and vertical stress, either equal to σ_1 (extensional regime) σ_2 (strike-slip regime) or σ_3 (compression regime). The common occurrence of anthropogenic seismicity during reservoir filling and earthquake-triggered earthquakes suggests that the continental crust is globally in a state of frictional equilibrium.

Applications to geological structures

Very little is known concerning the stress fields that exist in rocks during deformation, although it is one of the prime goals of the subject to define these fields as closely as possible. The lack of knowledge is, in part, due to the complexity of the stress fields that exist in deforming bodies but mostly results from an overall lack of information concerning the mechanical properties of rocks.
The application of normal and shear stresses can be illustrated with reference to two simple geological examples: the stress at a fault plane and the stress at a bedding plane undergoing flexural slip folding resulting from opposed compressive forces. The sense of fault displacement and bedding-plane slip can be predicted if the direction of the force is known, and vice versa.

Conclusion

The kinematic analysis identifies four components of deformation:
- Translation (change in position)
- Rotation (change in orientation)
- Dilation (change in size) and
- Distortion (change in shape).

Stress is an instantaneous quantity defined as a force per unit area. The stresses at a point are the vector components of the stress vectors on the three planes of reference. Mathematically, the expression of stress combines traction vectors for all possible planes associated with the point: This collection of forces per unit area is a tensor quantity.
The stress tensor σ_{ij} is a mathematical structure that describes the relation between two linked vectors: the force vector and the plane orientation vector. The first subscript indicates the direction of the force, while the second subscript denotes the face of the cube on which it is acting.
It requires nine numbers and a coordinate system to be defined: it is a second-order tensor. The 9 components of the stress tensor are the nine components of stresses. Stresses cannot be summed using vector addition.
The state of stress at a point is described by the magnitudes and orientations of the three principal stresses or the normal and shear stresses on a plane of known orientation. It can be represented by an ellipsoid with axial lengths determined by the three principal stresses.
The Mohr construction relates stress points to material planes. The stress state σ_S on planes of varying orientation describes a circle passing through σ_1 and σ_3 on the Mohr diagram, on which the magnitudes and orientations of shear stress σ_S as a function of the normal stress σ_N can be visualized.
Stresses in the lithosphere have both tectonic (plate motion, burial, and unburial, as well as magma intrusion) and non-tectonic, local origins (thermal expansion and contraction, meteoritic impacts, and fluid circulations). The regional uniformity of natural stress fields suggests a dominantly tectonic origin. Stress controls deformation. Therefore, understanding stresses is essential to describe, quantify, and predict rock deformation and tectonic processes.

Recommended literature

Hubbert M.K. & Rubey W.W. - 1959. Role of fluid pressure in mechanics of overthrust faulting. I. Mechanics of fluid-filled porous solids and its application to overthrust faulting. *Geological Society of America Bulletin* **70** **(2)**, 115-166.

Jaeger J.C. - 1969. *Elasticity, fracture and flow: with engineering and geological applications*, third ed. Methuen & Co LTD and Science Paperback, London, 268 p.

Jaeger J.C. & Cook N.G.W. - 1979. *Fundamentals of rock mechanics*, 3 ed. Chapman and Hall, London, 593 p.

Jaeger J.C., Cook N.G.W. & Zimmerman R.W. - 2007. *Fundamentals of rock mechanics. Fourth edition.* Blackwell Publishing, Oxford, 475 p.

Means W.D. - 1976. *Stress and strain. Basic concepts of continuum mechanics for geologists*. Springer Verlag, New York, 339 p.

Oertel G.F.M. - 1996. *Stress and deformation: a handbook on tensors in geology*. Oxford University Press, New York, 292 p.

Pollard D.D. & Fletcher R.C. - 2005. *Fundamentals of structural geology*. Cambridge University Press, Cambridge, 500 p.