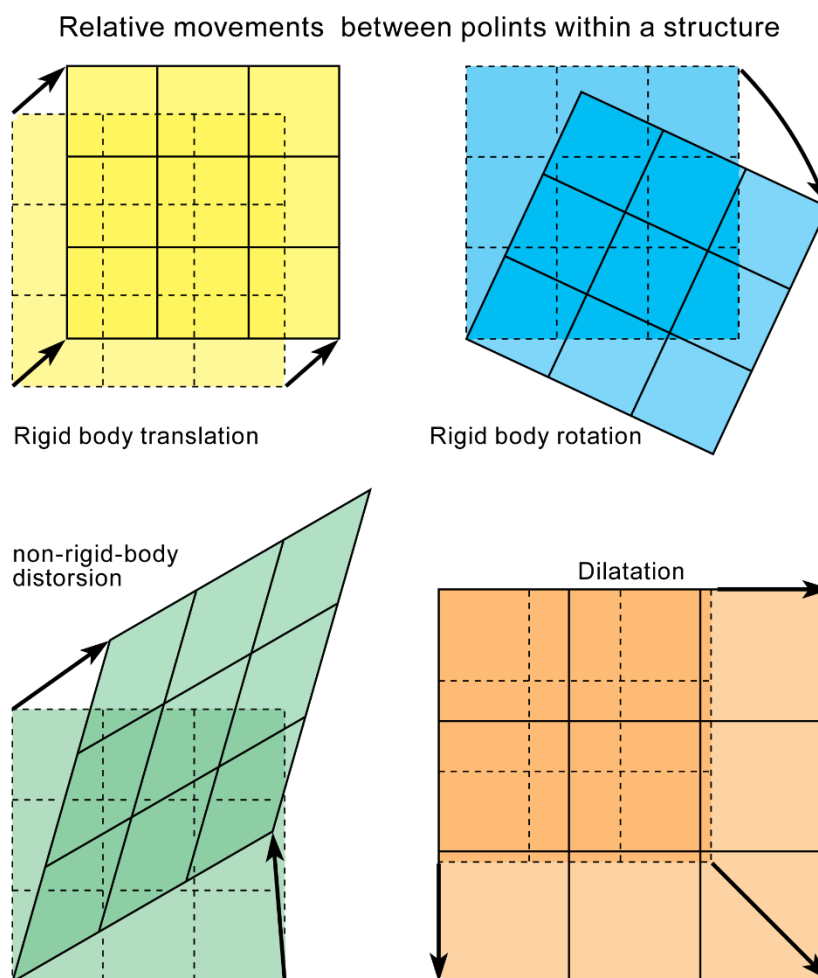


## Ductile deformation - Concepts of finite strain

Deformation includes any process that results in a change in shape, size or location of a body. A solid body subjected to external **forces** tends to move or change its displacement. These displacements can involve four distinct component patterns:

- 1) A body is forced to change its position; it undergoes **translation**.
- 2) A body is forced to change its orientation; it undergoes **rotation**.
- 3) A body is forced to change size; it undergoes **dilatation**.
- 4) A body is forced to change shape; it undergoes **distortion**.



These movement components are often described in terms of **slip** or **flow**. The distinction is scale-dependent, slip describing movement on a discrete plane, whereas flow is a penetrative movement that involves the whole of the rock.

The four basic movements may be combined.

- During **rigid body deformation**, rocks are translated and/or rotated but the original size and shape are preserved.
- If instead of moving, the body absorbs some or all the forces, it becomes **stressed**. The forces then cause particle displacement within the body so that the body changes its shape and/or size; it becomes **deformed**. **Deformation** describes the complete transformation from the initial to the final geometry and location of a body. Deformation produces discontinuities in brittle rocks. In ductile rocks, deformation is macroscopically continuous, distributed within the mass of the rock. Instead, brittle deformation essentially involves relative movements between undeformed (but displaced) blocks.

**Strain** describes the **non-rigid body** deformation, i.e. the amount of movement caused by stresses between parts of a body. Therefore, stresses and strains are interdependent. Particle displacements produce **dilatation** (change in size, positive for expansion and negative for shrinking) and/or **distortion**, a change in shape. The final shape, after cumulative strain(s), is what geologists can observe. The strain is a geometric concept intended to quantify the relative displacement and subsequent change in the configuration of the particles in a given dimension of the body, i.e. measuring strain is quantifying deformation from an “initial” shape to a “final” shape (both initial and final may be intermediate steps of a longer deformation evolution). A **strain analysis** consists in quantifying the changes in shape and size due to deformation. Thus, strain analysis involves determining the (1) strain orientation, (2) strain magnitude, and (3) patterns of strain variation. The **finite strain** is the measurable parameter that assigns a quantity to the total change in the shape of a deformed object compared to its original shape. Hence, the strain is a dimensionless measure of the amount of flow. This information helps in understanding the physical displacements that produced structures found in the field. In practice, dilatation is very difficult to measure so that geologists usually speak of strain for distortion only.

## Strain analysis

### Homogeneous - heterogeneous strain

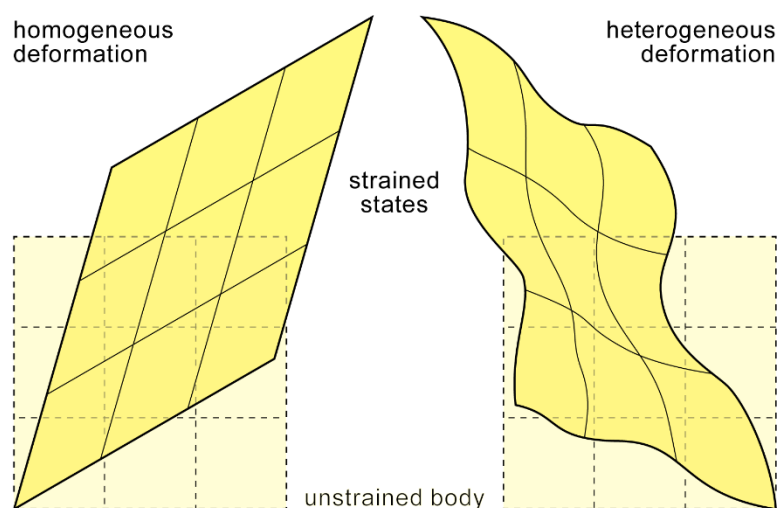
The concept of finite strain is useful for uniform and homogeneous deformation.

#### **Definition**

The strain is homogeneous if all parts of a body suffer distortion and/or dilatation characterized by the same amount, type and direction of displacement. Standard criteria are that:

- Straight and parallel lines and planes remain straight and parallel.
- All lines in the same direction in a body undergo the same extension and rotation.
- Circles become ellipses. Spheres enclosed in the unstrained body become an ellipsoid.

The strain is heterogeneous if distortion and/or dilatation differ from place to place in the body. Straight lines and planes become curved and parallel lines and planes do not remain parallel; circles and spheres become complex, closed forms.



This definition emphasizes the important difference between strain and structure. Structures such as folds or veins would not exist without the presence of “layered” heterogeneities, i.e. the juxtaposition of dissimilar materials. One may reduce heterogeneous deformation to homogeneous domains by decreasing the size of the studied volume. A homogeneous deformation on one scale may be part of a heterogeneous deformation on a different scale. Heterogeneous strain distributed with different characteristics in different parts of a rock or rock mass is **partitioned**. Consequently, structures and

strain refer to different reference frames, possibly geographical for structures, local for strain: the **fabric** of the rock, which includes foliation and lineation.

### *Continuum assumption*

Strain compatibility is a requirement for the rock to remain coherent during deformation. However, lines and planes may also be broken during heterogeneous and discontinuous deformation. Strain variations can be due to the different mechanical behaviors of components of a rock or association of rocks. For example, the presence of relatively rigid components can result in more intense strain concentrated into weaker parts of the rock rather than being evenly distributed. Strain analysis in rocks is essentially dedicated to homogeneous and continuous deformation, on the considered scale.

### Progressive deformation until the finite strain

**Progressive deformation** refers to the series of movements that affect a body from its initial undeformed stage to its deformed state. Progressive deformation can be continuous or discontinuous.

### *Deformation path*

The **state of strain** of an object is the total strain acquired by this object up to the time of measurement, i.e. the sum of all different shapes and positions it has undergone. The sequence of strain states through which the object passed during progressive deformation defines the **strain path**. Deformation paths follow successive stages during straining and are expressed relative to an external coordinate system. Each step that can be identified, i.e. each tiny division of the deformation path is an **increment**. The amount of deformation that occurs from one stage to the next is an **incremental strain**.

### *Infinitesimal strain*

Each increment can be divided into a series of smaller and smaller increments. When the duration tends to zero, the extremely small amount of strain is termed **infinitesimal**, also described as **instantaneous**. We will see that the orientation of instantaneous strain axes may be very different from that of finite strain axes.

### *Strain history*

The strain history is the sum of many strain increments, each of which having infinitesimal (instantaneous) characteristics. It is impossible to reconstruct if the strain is perfectly homogeneous. Rock masses are heterogeneous, so that different parts register different parts of the strain history. The challenge is to combine these parts to understand the whole deformation evolution.

### *Finite strain*

The sum of incremental strains, i.e. the total strain is the **finite strain**.

Although all states of strain result from progressive deformation, the finite strain does not provide any information about the particular strain path that the body has experienced. A variety of strain paths may lead to the same finite strain.

## **Measurement of strain**

If an object is deformed, the deformation magnitude depends on the size of the object as well as on the magnitude of the applied stress. To avoid this scale-dependence, the strain is measured as normalized displacements expressed in scale-independent, dimensionless terms. As a change in shape, the strain may be measured with two quantities:

- The change in length of lines: the linear strain or **extension**  $\epsilon$ .
- The change in angle between two lines or a line and a plane: the angular strain or **shear strain**  $\gamma$ .

Any strain geometry can be measured as a combination of these changes. The corresponding terms are defined as follows.

### Length changes

Two parameters describe changes in length: longitudinal strain and stretch.

### Longitudinal strain

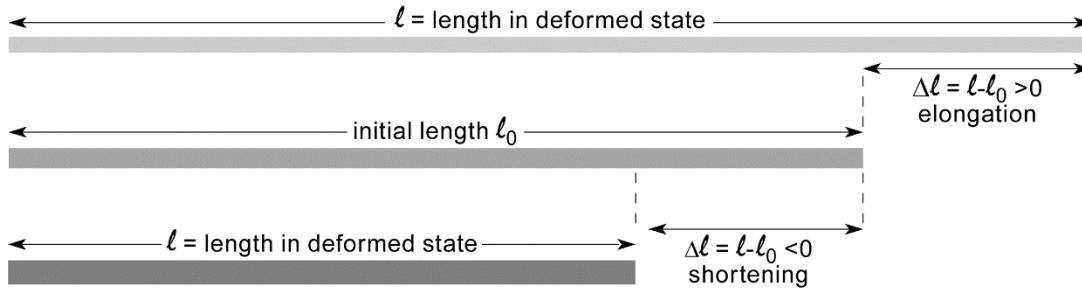
The **extension**  $\varepsilon$  is given by the dimensionless ratio:

$$\varepsilon = (\ell - \ell_0) / \ell_0 = (\ell / \ell_0) - 1 \quad (1)$$

Where  $\ell_0$  is the original length and  $\ell$  the new length of a reference linear segment in a rock mass.

If  $\ell > \ell_0 \rightarrow \varepsilon > 0$  A positive value indicates **elongation**.

If  $\ell < \ell_0 \rightarrow \varepsilon < 0$  A negative value indicates **shortening**.



Definition of the longitudinal strain  $\varepsilon = \Delta\ell / \ell_0$

This measure of how many percents a line gets longer or shorter is also termed **engineering strain**, in particular for laboratory experiments.

For an infinitesimal strain, we consider the one-dimensional deformation of an extensible axis  $Ox$ .  $P$  is a point at  $x_P$  from the origin  $O$ . When stretching the axis  $P$  comes to  $P'$  and  $OP' = x_P + \Delta x$ , with  $\Delta x$  being a linear function of  $x$  if stretching is homogeneous. Taking  $P$  very close to  $O$ , equation (1) becomes:

$$\varepsilon = [(x + \Delta x) - x] / x = \Delta x / x$$

The infinitesimal deformation at  $P$  is per definition

$$\varepsilon = \lim_{\Delta x \rightarrow 0} \left( \frac{\Delta x}{x} \right)$$

which is commonly written:

$$\varepsilon = d\ell / \ell \quad (2)$$

### Stretch

For large-scale deformations, the change in length of a line is generally given by the **stretch**  $S$ , which is the deformed length as a proportion of the undeformed length:

$$S = \ell / \ell_0 = 1 + \varepsilon \quad (3)$$

The stretch is also used to define the **natural strain**, which is the logarithmic strain  $= \ln(S)$ .

### Quadratic elongation

The **quadratic elongation**  $\lambda$  is the standard measure of finite longitudinal strain. It is the square of the stretch:

$$\lambda = (\ell / \ell_0)^2 = (1 + \varepsilon)^2 \quad (4)$$

Zero strain is given by  $\lambda = 1$ .

The **reciprocal** quadratic elongation  $\lambda' = 1/\lambda$  is used sometimes.

## Angular changes - Shear strain

### **Angular shear**

The angle  $\psi$  called **angular shear** is the change of an original right angle (i.e. the amount a vertical line has been rotated with respect to a horizontal line):

$$\psi = 90^\circ - (\alpha) \quad (5)$$

The sense of shear is anticlockwise or clockwise depending on the sense of angular change.  $\alpha$  is measured between two initially perpendicular lines. Therefore  $-90^\circ \leq \psi \leq +90^\circ$

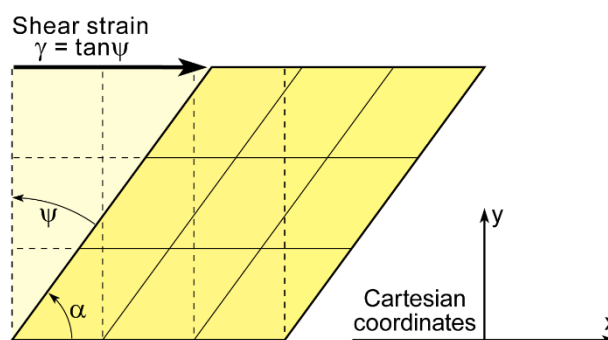
ATTENTION: anticlockwise and clockwise means sinistral and dextral, respectively, only and only when shearing refers to strike-slip movements. Otherwise, the sense depends on the direction of observation. It is then safer to specify top-to--.

The difficulty in rocks is to identify two lines that are known to be initially orthogonal.

### **Shear strain**

The **shear strain**  $\gamma$  is the tangent to the angular shear:

$$\gamma = \tan \psi \quad (6)$$



The tangent of a very small angle is equal to the angle in radians so that for infinitesimal strain  $\psi = \gamma$ .

In Cartesian coordinates  $\gamma = \Delta x / y$ . This equation shows that shear strain involves an interaction between the coordinate axes.

## Area - Volume change

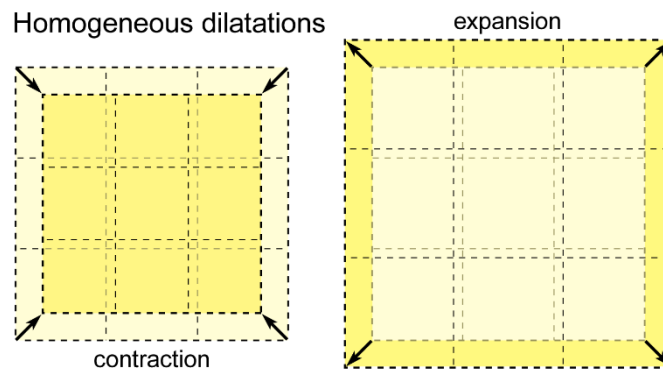
The volume change (**dilatation**) is given by:

$$\Delta = (V - V_0) / V_0 \quad (7)$$

where  $V$  and  $V_0$  are the volumes in the deformed and undeformed states, respectively. In two-dimensions, volume  $V$  is reduced to area  $A$ .

There is **expansion** if  $V > V_0$ .

There is **contraction** if  $V < V_0$ .



For infinitesimal strain  $\Delta_A = dA/A$  and, in three dimensions  $\Delta_V = dV/V$ .

### Strain analysis

The strain analysis consists either in finding the longitudinal and shear strains where principal strain axes are known, or inversely in finding the principal strain axes where the longitudinal and shear strains can be measured for various directions. The latter option, using deformed objects, is most common in geology.

### Graphical representation

A diversity of methods has been designed to estimate finite strain in **tectonites** (rocks whose fabric reflects the deformation history). Using mostly the distortion of objects (e.g. ooids and fossils) or point distributions (e.g. quartz grain centers in quartzites) all methods try to estimate the shape and orientation of the strain ellipse/ellipsoid.

### Strain ellipse

For the sake of clarity, we first think in two dimensions.

#### **Definition**

A circle of unit radius (but it can be of any size) and flattened vertically parallel to the coordinate axis is homogeneously deformed into an ellipse with two major axes which, initially, were diameters of the circle. This **strain ellipse** is a two-dimensional, graphical concept to visualize the amount of linear and angular strain involved in the deformation of rock. The radius of this ellipse is proportional to the stretch in any direction. Its longest and shortest radii, known as the **principal strain axes**, define the strain ellipse. In the considered coordinate-parallel flattening, these axes are vertical (short) and horizontal (long).

Three numbers only define the two-dimensional strain of the plane in which the ellipse lies:

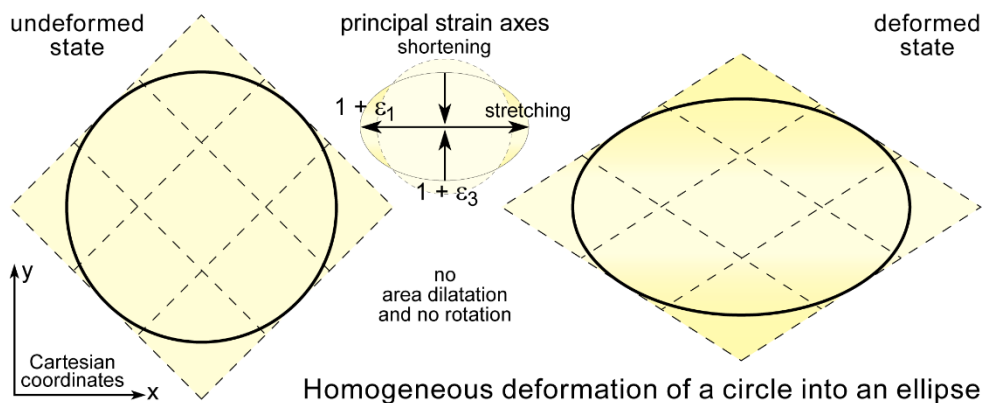
- The dimensions of the strain ellipse.
- The orientation of the principal strain axes and

From equations (3) and (4), saving the subscript 2 for the intermediate axis of the three-dimensional case:

The length of the long principal strain axis, the **stretch axis**, is  $1 + \varepsilon_1 = \sqrt{\lambda_1}$

The length of the short principal strain axis, the **shortening axis**, is  $1 + \varepsilon_3 = \sqrt{\lambda_3}$ .

The orientation is usually the anticlockwise angle between the abscissa and the longest principal strain axis.



There are two strain ellipses at any stage in the deformation: the **finite strain ellipse** represents the cumulative, total deformation; the **infinitesimal ellipse** represents the strain for an instant in time. In two dimensions, the **infinitesimal stretching axes** are the instantaneous direction of maximum elongation and the instantaneous direction of maximum shortening. They are therefore the lines that experience the fastest and slowest stretching rates of all possible line orientations at the considered increment.

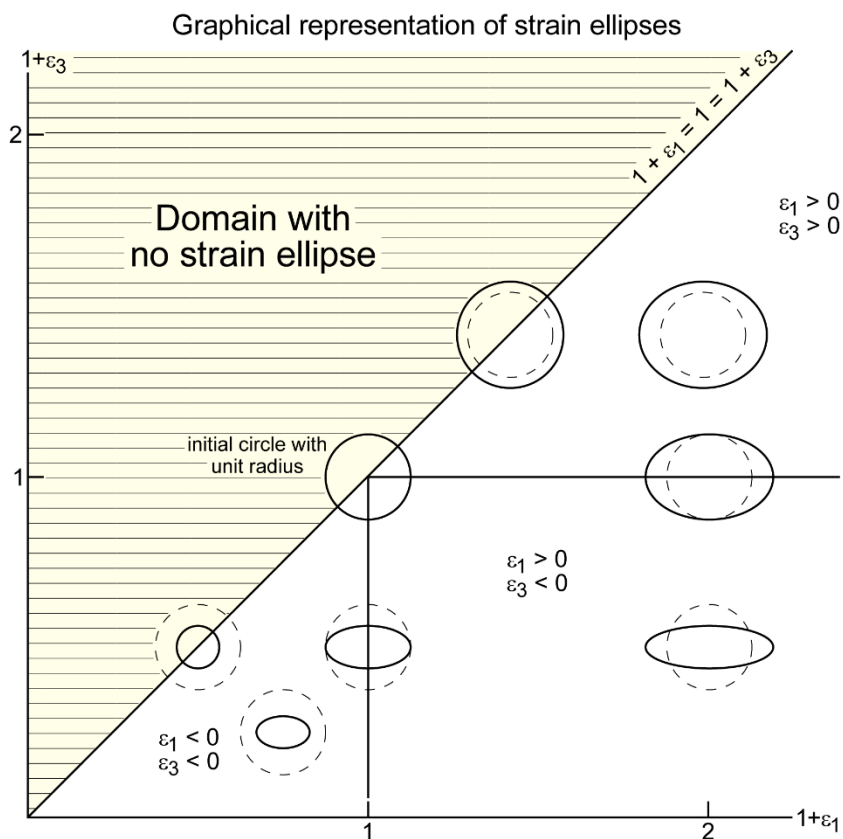
In all cases the strain axes are perpendicular. They are also lines of zero shear strain.

**Shape**

The shape of the ellipse is the ratio of the principal axes:

$$R = (1 + \epsilon_1) / (1 + \epsilon_3) \tag{8}$$

It embodies the overall intensity of distortion. A graphical representation consists in plotting  $1 + \epsilon_1$  along the horizontal axis and  $1 + \epsilon_3$  along the vertical axis of a scaled coordinate system.



The diagonal line is the locus of ellipses that remained as circles because they have undergone equal elongation or equal contraction in all directions, i.e.  $1 + \varepsilon_1 = 1 = \varepsilon_3$ . No ellipse plots above this line because  $1 + \varepsilon_1$  is always greater or equal to  $1 = \varepsilon_3$ . Two more particular lines can be drawn: one vertical for which  $\varepsilon_1 = 0$  and one horizontal for which  $\varepsilon_3 = 0$ . These two lines delimit three fields below the diagonal, where all possible strain ellipses are represented.

- The highest field (below the diagonal and above the horizontal line  $1 + \varepsilon_3 = 1$ ) includes ellipses in which both principal strains are positive elongations.
- The rectangular field below the horizontal line and to the right of the vertical line includes ellipses in which  $\varepsilon_1 > 0$  and  $\varepsilon_3 < 0$ .
- The triangle below the diagonal and to the left of the vertical line includes ellipses in which both  $\varepsilon_1$  and  $\varepsilon_3$  are negative shortenings.

### ***Dilatation***

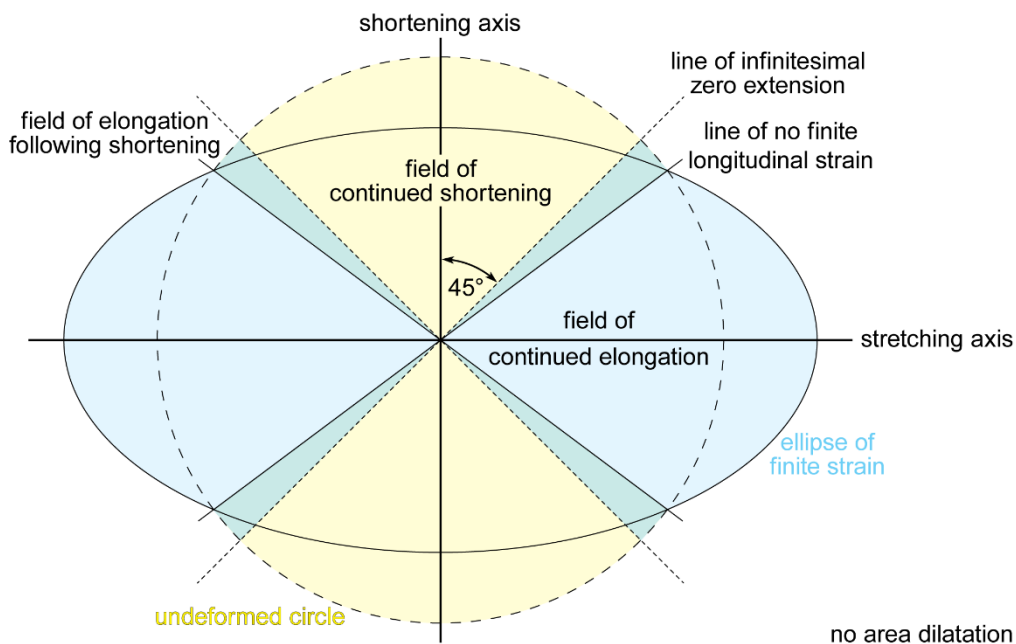
The area of the strain ellipse is the product of the principal axes.

$$1 + \Delta = (1 + \varepsilon_1)(1 + \varepsilon_3)$$

Dilatation  $\Delta$  is positive if the area increases, negative if it decreases.

### ***Change in length as a function of the orientation of a line***

The finite strain ellipse concentrically superimposed on the unstrained circle yields four intersection points. The two lines joining opposed points through the center have suffered no net change in length: those two are **the lines of no finite longitudinal strain**. By symmetry, the finite strain axes bisect these lines. All lines in the two sectors containing the longest strain axis increased in length, all lines in the sectors containing the shortest strain axis shortened.



Changing fields of line shortening/elongation from infinitesimal to finite strain in two dimensions

For the first infinitesimally small increment of the considered vertical flattening, the two lines of no incremental longitudinal strain are inclined at  $45^\circ$  to the strain axes. With further flattening, the two material lines of initial zero longitudinal strain turn towards the horizontal axis while the strain ellipse lengthens along this axis and flattens along the vertical axis. Yet, at any given time during deformation, lines within  $45^\circ$  of the horizontal axis are lengthening and lines within  $45^\circ$  of the vertical axis are shortening. Therefore, some material in the strain ellipse migrates through lines separating



fields of shortening and lengthening as the ellipse flattens. The strain ellipse is consequently divided into fields in which material lines have had different histories:

- Within  $45^\circ$  to the vertical axis, lines have been shortened throughout their deformation history.
- Between the  $45^\circ$  line and the rotated initial lines of no finite longitudinal strain, material lines have undergone an initial shortening, followed by lengthening.
- Between the lines of no finite longitudinal strain, symmetrically with respect to the horizontal stretch axis, lines have been permanently elongated.

Note that for extremely large strain, these lines will tend towards fusion with the long axis. Besides, lines of no finite longitudinal strain do not exist if the strain involves negative or positive dilatation because then all lines have negative or positive stretch, respectively.

### ***Application***

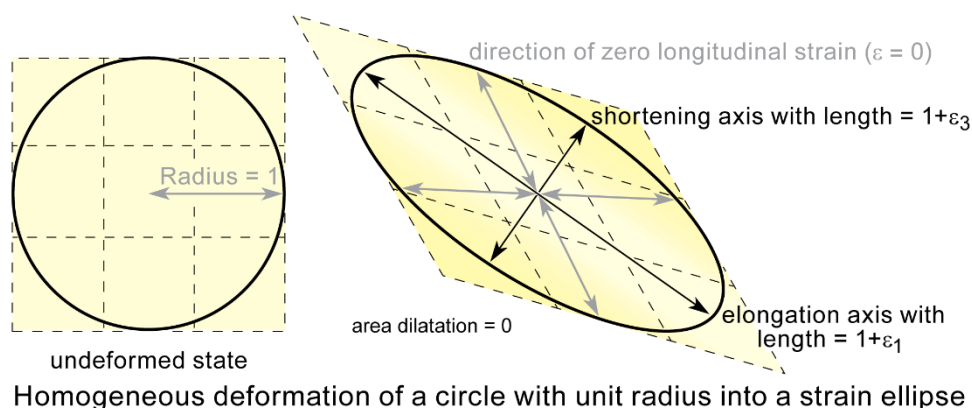
If an initially circular structure becomes elliptical after deformation, the two main axes  $k(1+\epsilon_1)$  and  $k(1+\epsilon_2)$  can be directly measured.  $k$  is a constant, since the size of the original circle is unknown. The slope of the regression line through many long axis / short axis measurements gives the ellipticity  $R$  of the finite strain ellipse.

### ***Rotational component of deformation***

The only rotation considered in two-dimensions is around an axis perpendicular to the plane of observation, hence perpendicular to the two strain axes.

The rotational part of deformation  $\omega$  is the angular difference in orientation of the strain axes with respect to a reference line. One commonly uses symbols as  $\theta$  and  $\theta'$  for orientation angles before and after deformation, respectively.

$$\omega = \theta' - \theta$$



### **Strain in three-dimensions: strain ellipsoid**

We use and derive in three dimensions measures of strain analogous to those in two dimensions.

#### ***Definition***

In a ductilely deforming medium, the material flows from the high-stressed to the low-stressed domains. At a scale at which deformation can be considered to be continuous and homogeneous, natural deformation in three dimensions is described as the change in the shape of an imaginary or material sphere with a unit radius  $r = 1$ . This unit sphere becomes an ellipsoid whose shape and orientation describe the strain. The equation describing this ellipsoid is:

$$\frac{x^2}{(1+\epsilon_1)^2} + \frac{y^2}{(1+\epsilon_2)^2} + \frac{z^2}{(1+\epsilon_3)^2} = 1$$

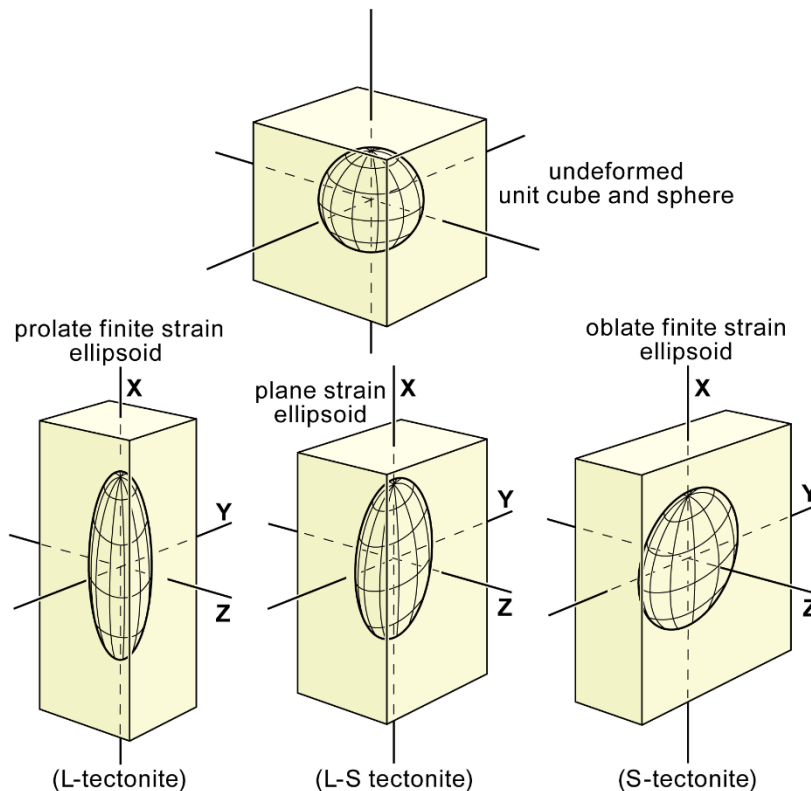
The three principal axes of this **finite strain ellipsoid** are the maximum, intermediate and minimum **principal strain axes** designated by X, Y and Z, respectively when lengths can be measured to define the strain magnitude. From equations (3) and (4), these lengths are:

$$X = 1 + \varepsilon_1 = \sqrt{\lambda_1} \quad \geq \quad Y = 1 + \varepsilon_2 = \sqrt{\lambda_2} \quad \geq \quad Z = 1 + \varepsilon_3 = \sqrt{\lambda_3}$$

respectively, since the original sphere has a radius of 1. The longest axis X is the stretching axis, Y the intermediate axis, and Z the shortening axis. The shear strain is zero in the direction of these three axes. Deformation in which linear extension of the three principal strain axes is non-zero ( $X \geq Y \geq Z$ ) is **triaxial**. Planes that contain two of these axes are **principal strain planes**.

The strain ellipsoid is the visualization of the second-order strain tensor.

Types of homogeneous deformation of a cube containing a sphere



The lines of no finite longitudinal strain in the strain ellipse correspond to two planes of no finite longitudinal strain (no stretching, no rotation) in the strain ellipsoid.

Cross-sections of the strain ellipsoid are ellipses (but there can be circular cross-sections). In general, the way we determine the 3D strain ellipsoid is to saw up rocks to define surfaces on which we find 2D strain ellipses. Then recombine the ellipses into an ellipsoid.

### Shapes

A single number, the strain ratio, specifies the shape (distortion component) of the strain ellipse in two dimensions. In three dimensions, two ratios of the principal strain axes fully characterize the shape of an ellipsoid: The ratio  $R_{1,2}$  of the longest and intermediate axes and the ratio  $R_{2,3}$  between the intermediate and shortest axes:

$$R_{1,2} = X/Y \quad \text{and} \quad R_{2,3} = Y/Z$$

The different shapes of finite strain ellipsoids can then be distinguished using the value:

$$K = (R_{1,2} - 1) / (R_{2,3} - 1)$$

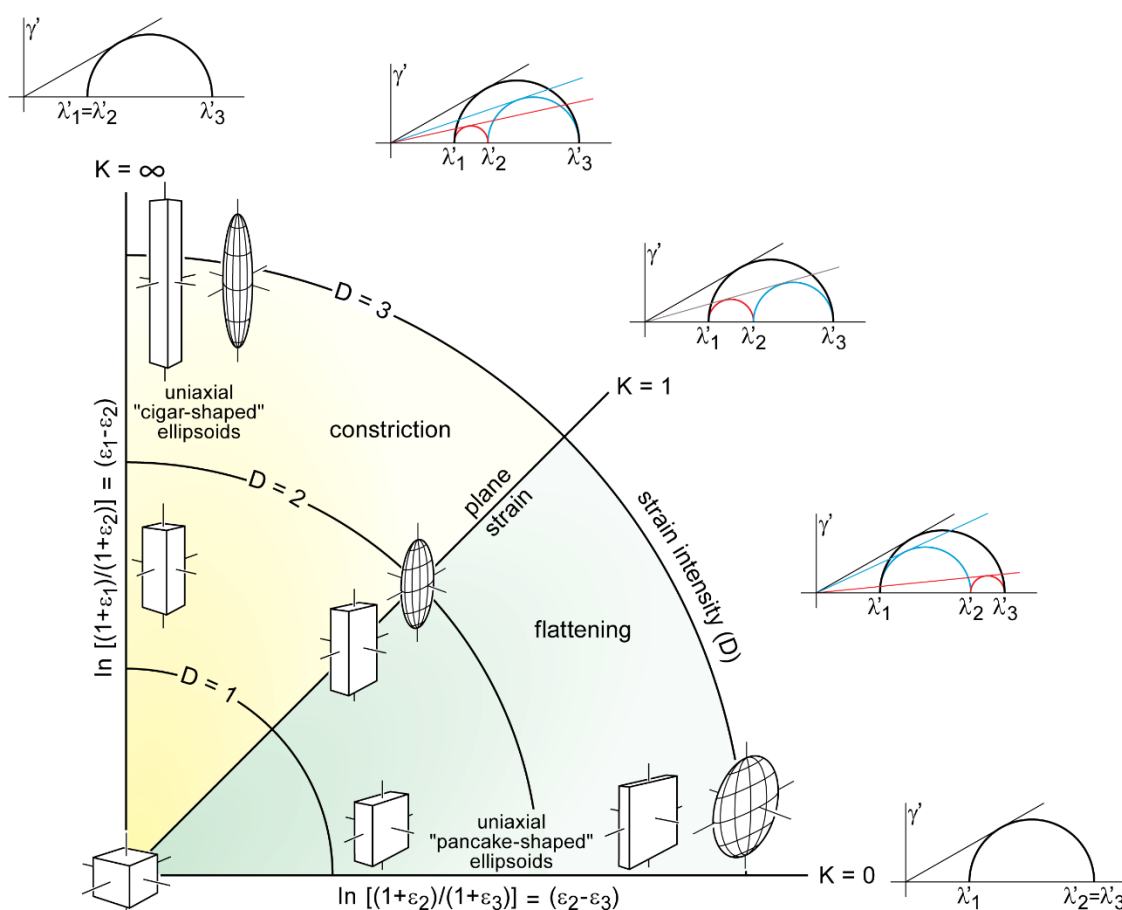
Notice that 1.0 is the minimum value of  $R_{1,2}$  and  $R_{2,3}$ .

Every ellipsoid can be reported as a point in the so-called Flinn diagram, where  $R_{1,2}$  is plotted vertically against  $R_{2,3}$  and  $K$  values describe slopes of lines passing through the origin.

The diagonal ( $R_{1,2} = R_{2,3}$ ) defines **plane strain** deformation where the Y-axis of finite strain ellipsoids keeps constant during deformation (i.e., there is neither extension nor shortening along the Y direction and all displacements due to deformation occur in the XZ plane).

The plain strain diagonal line divides the graph into two fields. At constant volume, the various strain states are:

- $K = \infty$  :            **Axially symmetric stretching**;  $X \gg Y \geq Z$ ; Ellipsoids have a long, cigar shape and plot near the vertical axis.
- $\infty > K > 1$  :        **Constriction strain**; in this case,  $X > Y = Z$ ; the ellipsoids have a prolate to cigar shape and plot near the vertical axis, above the plane strain line.
- $K = 1$  :                **Plane strain** at constant volume.
- $1 > K > 0$  :         **Flattening strain**; in this case,  $X = Y > Z$ ; the ellipsoids have an oblate to pancake shape and plot below the diagonal line, near the horizontal axis.
- $K = 0$  :                **Axially symmetric flattening**.  $X \geq Y \gg Z$ ; The flat ellipsoids plot along the horizontal axis.



Form of equal volume strain ellipsoids ( $K$ ) and degree of strain ( $D$ ) in the logarithmic Flinn diagram and corresponding three-dimensional strain Mohr circles with related tangents of maximum shear angle

### Volume change

Taking  $V = X.Y.Z$  and  $V_0 = 1$ , dilatation (equation 7) is expressed by:

$$\Delta + 1 = X.Y.Z = (1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3)$$

Plane strain conditions separating the constriction and flattening fields in the logarithmic Flinn diagram are defined as  $\epsilon_2 = 0$ . Rearranging in logarithmic terms yields the equation of a line with a unit slope. Volume changes produce a parallel shift of the plane strain diagonal line. If  $\Delta > 0$  the line intersects the vertical axis if  $\Delta < 0$  it cuts the horizontal axis.

**Strain intensity**

Several techniques provide a dimensionless measure of strain magnitude with reference to the shape of the strain ellipsoid, independently of its orientation.

The degree of strain visually increases away from the origin. It is defined in a logarithmic Flinn diagram as:

$$D = \sqrt{\log^2(R_{1,2}) + \log^2(R_{2,3})}$$

where  $\log(R_{1,2})$  and  $\log(R_{2,3})$  are the same ratios as defined for the Flinn diagram.

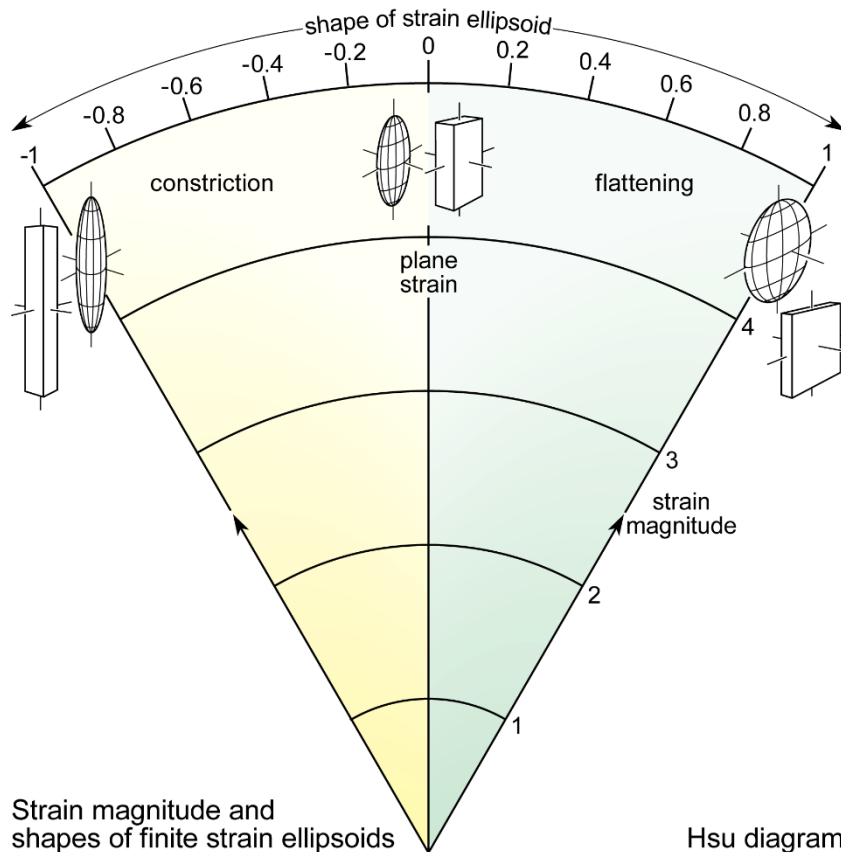
The so-called Hsu diagram uses natural strain. All strain states are represented on a 60° sector of a circle. The Lode's ratio  $-1 < v < +1$  defines the shape of the ellipsoids obtained:

$$v = (\log R_{2,3} - \log R_{1,2}) / (\log R_{2,3} + \log R_{1,2})$$

$v$  is measured along the circle arc. All plane strain ellipsoids (K line of Flinn diagram) plot along the bisecting arc  $v = 0$ , uniaxial constriction along the radius  $v = -1$  and all uniaxial flattening along the radius  $v = +1$ . The amount of strain is the distance from the circle center expressed by:

$$\bar{\epsilon} = (1/\sqrt{3}) \left[ (\log R_{1,2})^2 + (\log R_{2,3})^2 + (\log R_{1,3})^2 \right]^{1/2}$$

where  $R_{1,3}$  is  $Z/X$ .



Another measure also uses the ratios of principal strain axes to give intensity  $r = (X/Y) + (Y/Z) - 1$ .

**Note:** Strain ellipses and ellipsoids are shapes that offer no information about the deformation history; for example, identical shapes may result from simple or pure shear and the sum of two ellipses is another ellipse (ellipsoids in 3D). Conversely, superposed structures in anisotropic (layered or foliated) rocks that deformed heterogeneously may yield some relative chronology between deformation features.

### ***State of strain***

The orientation and magnitude of the three principal strain axes define the state of strain.

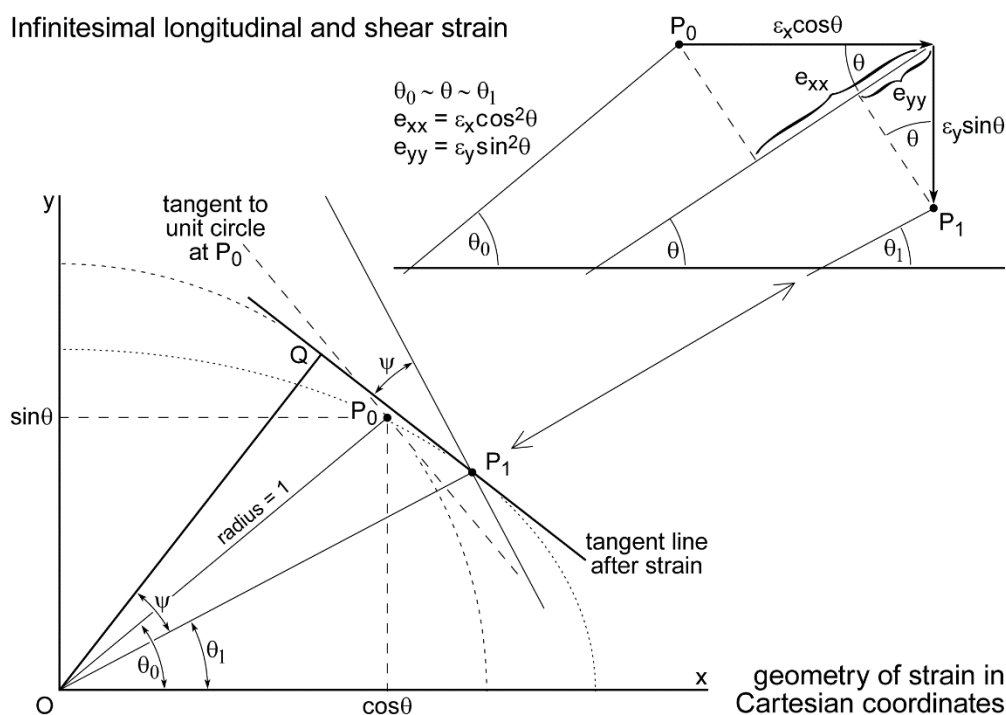
### ***Reciprocal strain ellipsoid***

The reciprocal strain ellipsoid is the ellipsoid of certain shape and orientation, contained within an unstrained body, which is transformed into a sphere by homogeneous deformation, i.e. the three principal axes are transformed into three orthogonal diameters of the sphere. The directions of the axes of the reciprocal strain ellipsoid are the original directions of the principal strain axes of the unstrained state.

### ***Mohr diagram for strain***

As previously, start considering a two-dimensional pure shear parallel to the coordinate axes  $x$  horizontal (elongation) and  $y$  vertical (shortening), crossing at a fixed origin  $O$ . Also, consider homogeneous and infinitesimal strain, so that linear and shear strains are so small that their products are negligible and all displacements are linear.

A point  $P_0$  has coordinates  $(x_0, y_0)$ . For convenience, consider the line  $OP_0$ , inclined at an angle  $\theta$  to the horizontal  $x$ -axis, to have unit length. Then  $x_0 = \cos \theta$  and  $y_0 = \sin \theta$ .



### ***Infinitesimal longitudinal strain***

After deformation,  $P_0$  has moved to  $P_1$  at coordinates  $(x_1, y_1)$ . The displacement  $P_0$  to  $P_1$  has two components, each parallel to the coordinate axes. They correspond to the linear extension (equation 1) of the horizontal and vertical lines defining  $x_0$  and  $y_0$ , respectively. These two components are:

Horizontal extension:  $\epsilon_x \cos \theta$

Finite strain

jpb, 2019

Vertical extension:  $\varepsilon_y \sin \theta$

Lines  $OP_0$  and  $OP_1$  are nearly superposed and parallel if displacement components are infinitesimally small. A simple geometric construction and application of the Pythagora's theorem then shows that lengthening components of the line  $OP_0 = OP_1$  are  $e_{xx} = \varepsilon_x \cos^2 \theta$  and  $e_{yy} = \varepsilon_y \sin^2 \theta$  (negative in the sketched case), respectively.

Then, the linear extension of the unit length line is:

$$\varepsilon = \varepsilon_x \cos^2 \theta + \varepsilon_y \sin^2 \theta$$

For infinitesimal strain, this can be converted in terms of quadratic elongation (see equation 4):

$$\lambda = \lambda_x \cos^2 \theta + \lambda_y \sin^2 \theta \quad (9)$$

Re-writing with the trigonometric substitution to double angles  $\cos^2 \theta = (1 + \cos 2\theta)/2$  and  $\sin^2 \theta = (1 - \cos 2\theta)/2$  yields a longitudinal elongation:

$$\varepsilon = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} \cos 2\theta \quad (10)$$

### *Infinitesimal shear strain*

The small displacement of  $P_0$  to  $P_1$  involves also a change in the slope of the line from  $\tan \theta_0 = y_0/x_0$  to  $\tan \theta_1 = y_1/x_1$ . This line rotation implies shear strain as in equation (6).

The line  $OP_0$  is the radius of the circle deformed into the strain ellipse passing through  $P_1$ . The line tangent to the circle in  $P_0$  (hence initially orthogonal to  $OP_0$ ) becomes inclined to  $OP_1$ . The change from right angle to the new angle is the angle of shear  $\psi$ . This can be visualized as the angle between the line orthogonal to  $OP_1$  and the inclined tangent to the unit circle, which is also the angle between  $OP_1$  and the normal to this inclined tangent. Call  $Q$  the point of intersection between the perpendicular from the origin to the distorted tangent and this line. Then the shear angle is given by:

$$\cos \psi = OQ/OP_1$$

which can be written with the secant function:

$$\sec \psi = OP_1/OQ$$

remembering the trigonometric function  $\tan^2 \psi = \sec^2 \psi - 1$ .

The standard equation of an ellipse whose major and minor axes coincide with the coordinate axes is:

$$(x/a)^2 + (y/b)^2 = 1$$

with  $a$  and  $b$  the semi-axes, in this case,  $1 + \varepsilon_x = \sqrt{\lambda_x}$  and  $1 + \varepsilon_y = \sqrt{\lambda_y}$ .

In coordinate geometry and assuming that for infinitesimal strain  $x^2 = x_0 x_1$  and  $y^2 = y_0 y_1$ , the equation of the ellipse at  $(x_1, y_1)$  is  $(xx_1/a^2) + (yy_1/b^2) = 1$

Since strain is very small, the equation of the ellipse simplifies to that of the tangent line at  $P_1$ . We express for  $P_1$ :  $x_1 = \cos \theta (1 + \varepsilon_x) = \cos \theta \sqrt{\lambda_x}$  and  $y_1 = \sin \theta (1 + \varepsilon_y) = \sin \theta \sqrt{\lambda_y}$

The tangent to the strain ellipse at  $P_1$  becomes:

$$(x \cos \theta / \sqrt{\lambda_x}) + (y \sin \theta / \sqrt{\lambda_y}) = 1$$

For infinitesimal strains, the variables can be taken as constants and we identify the standard equation of a line  $mx+ny+r=0$ .

The perpendicular distance of a line to the origin is  $d = r/\sqrt{m^2 + n^2}$

Where radius  $r$  is an absolute value.

Thus we can write 
$$OQ = \sqrt{1/\left[\left(\cos^2 \theta/\lambda_x\right) + \left(\sin^2 \theta/\lambda_y\right)\right]}$$

Hence 
$$\sec \psi = \sqrt{\lambda \left[\left(\cos^2 \theta/\lambda_x\right) + \left(\sin^2 \theta/\lambda_y\right)\right]}$$

Subtracting 1, we obtain the shear strain:  $\tan^2 \psi = \lambda \left[\left(\cos^2 \theta/\lambda_x\right) + \left(\sin^2 \theta/\lambda_y\right)\right] - 1$

which in equation (9) yields:

$$\gamma = \frac{(\lambda_x - \lambda_y)}{\sqrt{\lambda_x \lambda_y}} \cos \theta \sin \theta$$

Expressing 
$$\sqrt{\lambda_x \lambda_y} = \sqrt{(1 + \varepsilon_x)^2 (1 + \varepsilon_y)^2}$$

Infinitesimal strains become so small when they are squared that they can be neglected. Then  $\sqrt{\lambda_x \lambda_y} \approx \sqrt{1}$  and:

$$\gamma = (1 + \varepsilon_x)^2 (1 + \varepsilon_y)^2 \cos \theta \sin \theta$$

Using the double angle for single angles and developing the squared strain terms yields:

$$\frac{\gamma}{2} = \frac{(\varepsilon_x - \varepsilon_y)}{2} \sin 2\theta \quad (11)$$

### ***Mohr circle for infinitesimal strain***

Equation (10) and the halved equation (11) can be squared and added to write:

$$\left[\varepsilon - (1/2)(\varepsilon_x + \varepsilon_y)\right]^2 + (\gamma^2/2) = \left[(1/2)(\varepsilon_x - \varepsilon_y)\right]^2 (\cos^2 2\theta + \sin^2 2\theta)$$

Since  $\cos^2 + \sin^2 = 1$ , for any angle:

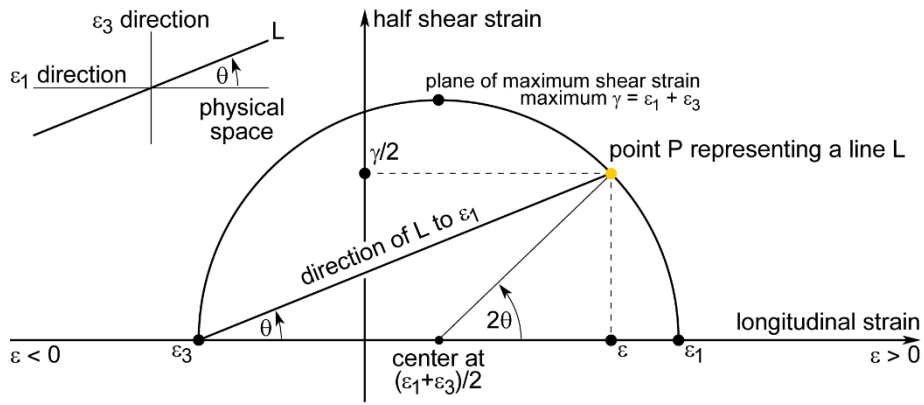
$$\left[\varepsilon - (1/2)(\varepsilon_x + \varepsilon_y)\right]^2 + (\gamma^2/2) = \left[(1/2)(\varepsilon_x - \varepsilon_y)\right]^2$$

This solution has the form of the standard equation of a circle in the coordinate plane (x,y) with center at (h,k) and radius  $r$ :

$$(x-h)^2 + (y-k)^2 = r^2$$

where  $x = \varepsilon$ ,  $y = \gamma/2$  and  $k = 0$ .

A Mohr construction is thus valid for an infinitesimal strain with normal extension as abscissa. In the circle of strain, the ordinate represents only one-half of the shear strain. Like for stresses, an angle subtended at the center of the Mohr strain circle by an arc connecting two points on the circle is twice the physical angle in the material. This readily shows that the maximum shear strain occurs on planes at  $45^\circ$  to the maximum elongation.



Two-dimensional Mohr construction for infinitesimal strain

### Mohr circle for finite strain

Approximations made for infinitesimal strain ( $P_0 = P_1$  and  $\theta_0 = \theta = \theta_1$ ) cannot be accepted for larger, geological and measurable strain.

Longitudinal strains are considered using the reciprocal quadratic elongation  $\lambda' = 1/\lambda = (1 + \varepsilon)^2$  (equation 4). The derivation for changes in length is similar to that for infinitesimal longitudinal strain, yielding a result comparable to equation (9) rewritten with  $\lambda'$  and  $\theta'$ , the orientation of a marker line after deformation:

$$\lambda' = \lambda'_x \cos^2 \theta' + \lambda'_y \sin^2 \theta' \quad (12)$$

A formulation resembling equation (10) for an infinitesimal longitudinal strain is obtained using the double angle identities  $\cos^2 \theta' = (1 + \cos 2\theta')/2$  and  $\sin^2 \theta' = (1 - \cos 2\theta')/2$ :

$$\lambda' = \frac{\lambda'_x + \lambda'_y}{2} + \frac{\lambda'_x - \lambda'_y}{2} \cos 2\theta' \quad (13)$$

Assessing shear strain starts, like for infinitesimal strain, with the equation of the ellipse, but now expressed in terms of reciprocal quadratic elongation:

$$\lambda'_x x'^2 + \lambda'_y y'^2 = 1$$

with  $x'$  and  $y'$  are the coordinates of a point (i.e. the directions of a radius vector) after deformation. The equation of a line tangent to the ellipse at that point derives from the equation of the ellipse:

$$\sqrt{\lambda'} = \left( \lambda'_x \cos \theta' \right) x' + \left( \lambda'_y \sin \theta' \right) y'$$

Normalizing, expanding, squaring, substituting and developing as detailed in specialized literature yields, somewhat expectedly, an equation similar to equation (11):

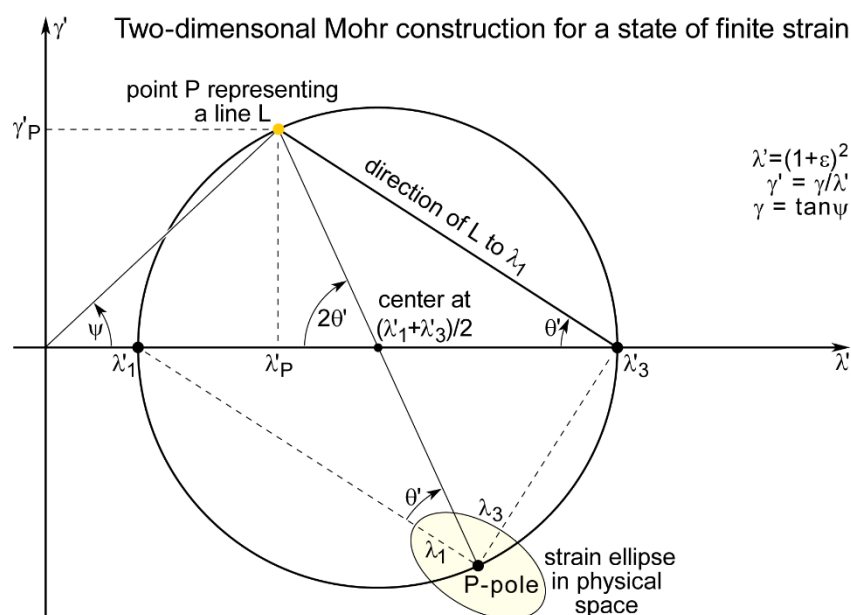
$$\gamma' = \frac{(\lambda'_x - \lambda'_y)}{2} \sin 2\theta' \quad (14)$$

where  $\gamma' = \gamma/\lambda'$ .

The two measures on the left sides of equations (13) and (14) plotted against each other define a circle, the Mohr diagram for the strained state, centered on the abscissa between  $\lambda'_x$  and  $\lambda'_y$ . Note that per definition  $\lambda'_x = 1/\lambda_x$  represents the greater principal strain but plots on smaller values than  $\lambda'_y = 1/\lambda_y$ , the smaller principal strain. The strain Mohr circle is entirely on the right side of the origin (the  $\lambda'$ -axis is horizontal) since the reciprocal quadratic elongation  $\lambda' = 1/\lambda = (1 + \varepsilon)^2$  is always positive.  $\gamma'$  is a measure of  $\gamma$ , the shear strain of a material line at  $\theta'$  to the identifiable  $\lambda_x$  direction,



in the deformed state. The lines of no finite elongation plot have  $\lambda' = 1$ . They are the intersections of the vertical line through this abscissa point and the circle.



There are important differences with the stress Mohr circle.

- The strain Mohr circle represents a two-dimensional state of homogeneous finite strain.
- Any point on the circle represents a line, not a plane as for the stress Mohr circle.
- Using reciprocal elongations, the strain angles  $\theta'$  are positive clockwise and laid off from the left side of the circle center.
- However, the shear angle  $\psi$  measured between the abscissa and the line connecting the origin to any strain point (since  $\gamma' = \gamma / \lambda'$ ) remains positive anticlockwise.

### *Mohr circles for infinitesimal and finite strain in three dimensions*

Like with stresses, the reciprocal intermediate principal strain axis can be plotted on the abscissa. Two circles within the defined two-dimensional circle represent the whole state of three-dimensional finite strain.

## Strain regime

As a first approximation, the general strain is assumed to be homogeneous and can be discussed in two dimensions (plane strain), where no area-change has taken place. The notions in two dimensions (2D) are then extended to three dimensions (3D). General deformation has two endmembers:

- Pure shear.
- Simple shear;

They refer to two types of deformation paths, respectively: coaxial and non-coaxial. These two terms refer to characteristic conditions under which deformation, as a process, occurs and evolves. Per definition, simple shear and pure shear are therefore strain regimes.

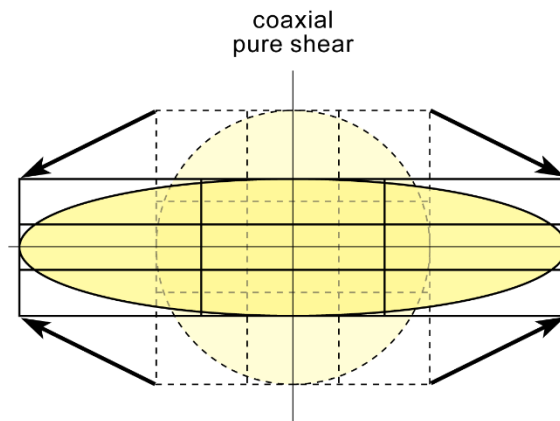
### Coaxial deformation; Pure shear

#### *Coaxial deformation path*

A **coaxial** deformation path is one in which the principal strain axes before and after strain coincide. They remain parallel to the same material lines throughout straining (i.e. the axes of the finite and infinitesimal strain ellipses remain parallel throughout the deformation). The **coaxial** deformation is **irrotational**.

In 2D, coaxial deformation transforms a square into a rectangle by homogeneous flattening. Two opposite sides of the square are shortened in one direction; the other two sides are elongated in the

orthogonal direction. After coaxial deformation, the sides of the square remain parallel and perpendicular.



In 3D, a cube of rock shortens in one direction and extends in the perpendicular direction. The cube is converted to a right parallelepiped while the sides of the cube remain parallel throughout the deformation, i.e. the sides of the final parallelepiped are parallel to the sides of the original cube. The orientation of the principal axes X, Y and Z does not change during progressive, homogeneous deformation.

#### *Pure shear*

If the Y-axis of the strain ellipsoid remains constant in length (plane strain), a coaxial strain may be fully described in the plane containing the X and Z axes, i.e. the 2D square transformed into a rectangle. If additionally, the area of the rectangle is the same as that of the initial square, the 2D constant area is a constant volume in 3D. Such a constant volume, coaxial and plane strain is **pure shear**. Note that all planes and lines (except the principal strain planes and axes) rotate towards the plane of maximum flattening and the line of maximum extension. In that case:

$$(1 + \varepsilon_1) = 1 / (1 + \varepsilon_3)$$

#### *Coaxial total strain ellipse*

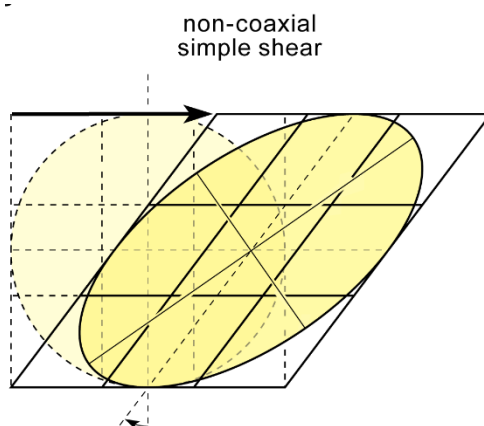
However, a coaxial strain may also include uniform dilatation.

#### *Non-coaxial deformation; simple shear*

##### *Non-coaxial deformation path*

A **non-coaxial** deformation path is one in which the directions of the principal strain axes rotate through different material lines at each infinitesimal strain increment: non-coaxial deformation is **rotational**. The axes of the finite and infinitesimal strain ellipses are not parallel.

In 2D, non-coaxial deformation transforms a square into a parallelogram in response to a shear couple. Two opposite sides of the square (traditionally the two vertical sides) are progressively inclined in one direction; the other two, traditionally horizontal sides are parallel to the shear couple and remain parallel.



In 3D, a cube of rock becomes a parallelepiped while the upper and lower sides of the cube remain parallel throughout the deformation. The principal axes X and Z rotate during progressive, homogeneous deformation. The intermediate strain axis Y remains parallel to itself; it is a stable direction throughout simple shear deformation.

### *Simple shear*

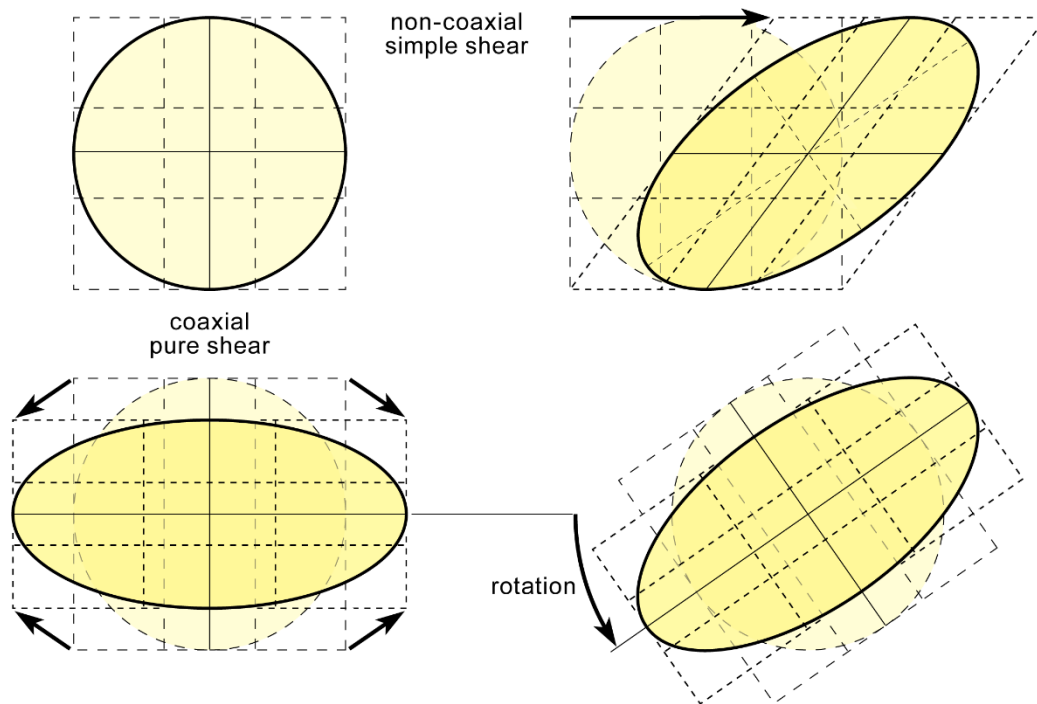
**Simple shear** is plane strain non-coaxial deformation. Volume remains constant and the Y-axis of the strain ellipsoid remains constant in length. Simple shear is analogous to the process occurring when a deck of cards is sheared to the right or left with each successively intervening and equally thin card sliding over its lower neighbor. Since the cards do not widen (their short side is parallel to the Y-axis of the strain ellipsoid), one can observe from the side what simple shear is in the plane containing the X- and Z-axes, with a particular interest in the deformation of material lines. A square (or the rectangle defined by the limits of the card deck) subjected to simple shear is converted into a parallelogram. The vertical sides of the square rotate but remain parallel to each other during deformation. These two sides progressively lengthen as deformation proceeds, but the top and bottom lines neither stretch nor shorten. Instead, they maintain their original length and remain parallel to each other. These stable lines, which are the card plane in three dimensions, are directions of no stretch (lines of no finite longitudinal strain). They represent the **shear direction**, while any card materializes the **shear plane**. All lines that are not parallel to the shear direction, on the shear plane, rotate in the same sense as the shear towards the shear plane and the line of shear direction.

Recall: - In three dimensions, the intermediate strain axis Y is a direction of no change in length that remains parallel to itself (the frontal side of cards). It is a stable direction throughout simple shear deformation.

- One line of no finite longitudinal strain is already defined: it is fixed and parallel to the shear plane; consequently, the fields of linear extension and shortening are asymmetrical.

### *Non-coaxial total strain ellipse*

Any arbitrary strain can be described in terms of a distortional component, which measures the ellipsoid shape plus a rotational component, which is the internal rotation of the principal strain axes from their original attitudes in the unstrained state.



### Coaxial / non-coaxial strain: ambiguity

A rotational strain is equivalent to a coaxial strain combined with external, rigid body rotation. Therefore, the shape of a finite strain ellipsoid cannot indicate how the strain was produced.

Since all planes and lines (except the principal ones) rotate towards planes of maximum flattening and lines of maximum extension with both coaxial and non-coaxial deformation, everything moves to the same plane and line in high strain deformation.

### Coaxial / non-coaxial strain: distinction

Particle motion: symmetrical for coaxial deformation, asymmetrical for non-coaxial deformation.

Instantaneous stretching axes: parallel and perpendicular to shear zone for coaxial deformation, inclined to shear zone for non-coaxial deformation.

Finite stretching axes: fixed for coaxial deformation, rotate for non-coaxial deformation

## **Strain markers**

Strain markers are any objects whose original shapes either are well known from undeformed rocks or can be estimated. Assuming that they deform passively within and with their matrix, their changed shapes reflect the intensity and the regime of strain in the rock. Several techniques have been devised for the measurement of strain from a variety of objects in rocks. The magnitude of strain is measured by comparing the final shape and configuration with the initial ones. Common shapes used in strain analysis include the sphere, circle, ellipse, bilaterally symmetric forms, and forms without bilateral symmetry. Accordingly, strain markers are grouped as follows:

Initially spherical objects (e.g. ooids, cylindrical worm burrows, reduction spots)

Initially ellipsoidal objects (e.g. pebbles, xenoliths)

Initially linear object (e.g. belemnites)

Objects with known angular features (e.g. fossils)

Evenly distributed objects (e.g. centers of minerals, pebbles).

Strain measurements are useful to reconstruct the regional strain field.

### Limitations and warning

#### ***Matrix-object relationship***

Not all deformed objects can be used as an absolute measure of strain because in some rocks the matrix deforms more easily than the strain markers. In this respect, most reliable strain markers are

worm tubes or reduction spots that have the same competence as the rock matrix, so that they change shape as much as the bulk rock.

### ***Orientation of plane of study***

Strain measurements are often dealing with

- changes in the ratio of one length relative to a length in another direction and
- the change in angle between two lines that were initially at a known angle.

Results are affected by the orientation of planes of study. In addition, the original size of any individual marker may be unknown.

### ***Volume changes***

The true strain is most easily measured if the deformation is volume constant. In this case, shortening in one direction is balanced by extension in the perpendicular direction (Poisson effect). The volumetric component of strain cannot be detected by relating an original shape to a final shape. A further complication arises if the volume reduction during deformation affects only the matrix without affecting the strain marker.

The volume change  $1 + \Delta_V$  is obtained from  $(1 + \varepsilon_1)(1 + \varepsilon_2)(1 + \varepsilon_3)$ .

### ***Initially spherical objects: Direct measurement of elliptical objects***

The simplest technique for measuring volume-constant strain uses initially spherical markers. These spheres are deformed into ellipsoids displaying elliptical cross-sections whose axes are also those of the strain ellipsoid. By combining data on axial ratios from differently oriented cross-sections, the three-dimensional volume-constant strain of the rock may be determined. In practice, the initial shape of spherical strain markers such as oolites is not perfectly spherical, and the deformation varies from point to point. Thus, a deformed oolitic limestone contains a variety of sizes and shapes of deformation ellipsoids, and many of them have to be measured to derive an average bulk strain for the rock.

### ***Initially cylindrical objects: Direct measurement of longitudinal strain***

Linear objects for which one can reconstruct the pre-deformation length are sometimes available. For example, rigid objects such as belemnite fossils and tourmaline crystals may undergo boudinage during elongation. The original length of the object can be determined by simply adding up the lengths of all fragments. The final length can be measured directly, and the stretch  $(\ell/\ell_0)$  can be calculated.

The assumptions for this method are:

- There is no distortion of the boudins
- The separation of the boudins represents the whole of the longitudinal strain.

If these assumptions are satisfied, this method can in theory yield both the dilational and distortional components of the strain ellipse.

Once several objects have been measured, their orientations and lengths can be plotted in radial coordinates. A best-fit ellipse may be estimated using a set of elliptical templates. In theory, a minimum of three points may constrain the ellipse but usually more are advisable.

Crinoid columns with circular cross-sections are also easy to use for bulk strain measurements. In most cases, the columnals were originally oblique to the bedding plane in which their cross-sections were thus elliptical. Strain techniques using initially elliptical objects as strain markers permit calculating strain in the bedding plane. Again, many initial sizes, shapes, and orientations make many measurements necessary for assessing a bulk three-dimensional strain.

### ***Objects with known angular features: Wellman methods and Mohr circle***

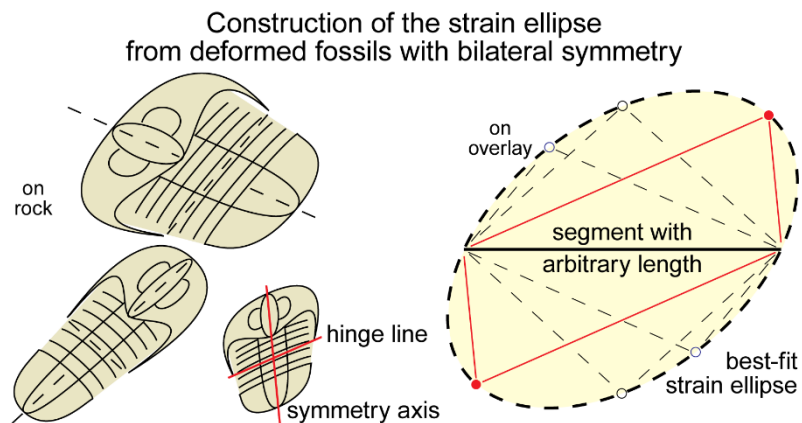
#### ***Deformed fossils with bilateral symmetry***

Fossils often possess bilateral symmetry, known angular relationships and proportion characteristics that typify a given species. Knowing their original shape, they may be used as strain markers provided

several (ideally >10) are available in the outcrop. An important assumption is that the fossils have deformed homogeneously with their matrix. Shear strain is deduced from the angular relationship of two originally orthogonal (symmetry and hinge or edge) lines; several fossils whose symmetry planes intersect the bedding plane in various orientations show various amounts of shear strain. This orientation-dependent effect is used to calculate volume-constant strain in the studied plane.

### Procedure

- Draw on tracing paper a horizontal reference segment of arbitrary length.
- For each fossil, trace the originally perpendicular symmetry and hinge lines, both starting at each end of the reference segment (thus two pairs of lines form a parallelogram).
- Mark the corners of the parallelogram.
- Repeat for each fossil.
- The marked points should fall on a strain ellipse with the reference line as a radius.



This (Wellman) method yields only the shape (distortion component) and not the size (dilation component) of the strain ellipse in the measurement plane.

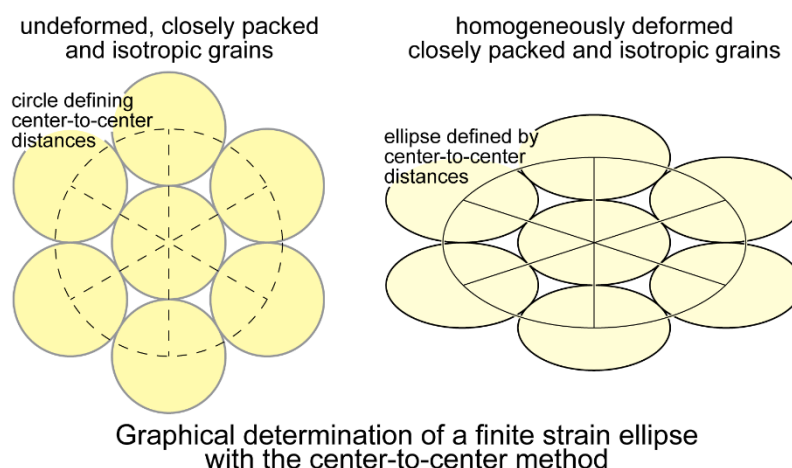
### *Deformed fossils without bilateral symmetry*

Fossils without bilateral symmetry are also useful for determining strain the ellipse of a deformed rock but the graphical constructions are complicated.

### *Mohr strain diagram*

#### *Fry method: Center-to-center distance*

Strain markers in rocks may be too strong to deform equally with their matrix. Examples include pebbles in a soft matrix, sand grains, ooids, feldspar porphyroclasts, etc. The shapes of these objects cannot be used to determine the strain. Instead, it is possible to use their center-point spacing if the objects were uniformly spaced in all directions before deformation (e.g. closely packed circular grains or ooids) and had no shape preferred orientation. Assume that their centers represent these undeformed objects. The pre-deformation and smallest center-to-center distance of two neighboring objects was equal in all directions if their size and distribution were homogeneous and isotropic, respectively. Then the minimum distances plotted in every direction between neighboring centers should delineate a circle around any grain center. Deformation changes this minimum distance. The pre-deformation circle becomes shortened or stretched by homogeneous deformation of the matrix along the axes of the strain ellipse. Therefore, the distances between centers of objects reflect the orientation and shape of the strain ellipse in 2D, ellipsoid in 3D.



An important assumption is that the strain markers did not slide past each other during deformation.

### Procedure

- Draw two orthogonal lines intersecting in the middle of a sheet of tracing paper. Place the origin over the center of one object.
- Mark the centers of all nearby objects.
- Shift the tracing paper (without rotation) to the next object. Repeat marking centers of all nearby objects.
- Repeat the procedure for all objects.
- After a sufficient number of objects has been used (typically >50), an elliptical vacancy appears around the reference intersection point. The shape of this vacancy is the shape of the strain ellipse.

### ***$R_f/\phi$ method: Deformed ellipsoidal objects***

The  $R_f/\phi$  method is a practical tool for strain determination from originally ellipsoidal objects such as pebbles. In a given rock section,  $R_f$  is the ratio of the long axis to the short axis of a deformed elliptical object. This ratio combines the initial ratio (ellipticity)  $R_i$  of the measured object and the axial ratio of the strain ellipse  $R_s$  in this rock section.  $\phi$  is the  $\pm 90^\circ$  angle (clockwise versus anticlockwise) between the long axis of the elliptical object and an arbitrary reference line. Assumptions are homogeneous deformation of equally deformable objects with initial random orientation and no ductility contrast with the matrix to avoid rotation components.

### Procedure

- Measure the angle  $\phi$  between the long axis of each elliptical object (>10, the more the better) and the arbitrary reference line.
- Measure the axial length of each object and calculate the shape ratios  $R_f$ .
- Plot each pair of values as a point on a graph  $R_f$  against  $\phi$ .

The points are scattered within a drop-shaped cloud. The position of  $R_f$ -max determines the orientation of the X-axis relative to the reference line; the Y-axis is perpendicular to the X-axis in the section surface. Standardized reference curves used as overlays on the data greatly facilitate the evaluation of the scatter of points in  $R_f/\phi$  plots.

## **Finite strain and displacement: Mathematical description**

Since deformation defines the change in position of points, one way to characterize any deformation is to assign a displacement vector to every point of a body placed in a Cartesian coordinate frame. The displacement vector joins the position of the considered point in its reference (generally initial) location to its position in the deformed situation. This is a forward problem, in which the variables

are material coordinates. One can also take spatial coordinates to try reconstructing the initial shape; this is an inverse problem.

### Displacement in 2-dimensions

#### **Coordinates**

The material point  $P_0$  with initial, material coordinates  $(x_0, y_0)$  has moved, after deformation, to  $P_1$  at spatial coordinates  $(x_1, y_1)$ . For homogeneous deformation of an incompressible body, the displacement of  $P_0$  to  $P_1$  is, mathematically, in two dimensions, a coordinate transformation. To get it, we write the table:

	$x_0$	$y_0$
$x_1$	$a$	$b$
$y_1$	$c$	$d$

and obtain equations:

$$x_1 = ax_0 + by_0$$

$$y_1 = cx_0 + dy_0$$

#### **Deformation matrix**

The mathematical basis for the 2D graphical representation of finite strain as strain ellipse and Mohr strain circle is a 2 x 2 matrix of numbers known as the deformation gradient tensor, simply the **deformation matrix**.

A vector is a one-column matrix. If a vector represents the coordinates of a point in space, then matrix multiplication may be used to transform that point to a new location:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

where the **deformation matrix** for infinitesimal or homogeneous deformation is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} \\ \epsilon_{yx} & \epsilon_{yy} \end{bmatrix}$$

This is written in a simplified form with the deformation matrix  $D$  in bold:

$$\mathbf{D}x_0 = x_i$$

Remember:

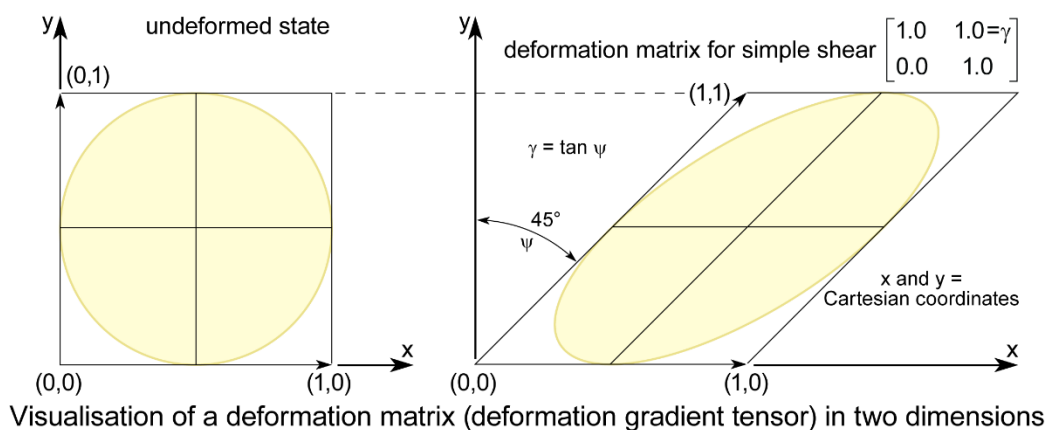
Matrix multiplication is as follows:

$$\begin{bmatrix} a_{xx} & a_{xy} \\ a_{yx} & a_{yy} \end{bmatrix} \begin{bmatrix} b_{xx} & b_{xy} \\ b_{yx} & b_{yy} \end{bmatrix} = \begin{bmatrix} a_{xx}b_{xx} + a_{xy}b_{yx} & a_{xx}b_{xy} + a_{xy}b_{yy} \\ a_{yx}b_{xx} + a_{yy}b_{yx} & a_{yx}b_{xy} + a_{yy}b_{yy} \end{bmatrix}$$

Like for stress notation, the first subscript means “act on a plane orthogonal to subscript axis”, the second subscript means “oriented parallel to subscript axis”.

In general, we can visualize the type of strain represented by a deformation matrix by imagining its effect on a unit square. The two columns of the deformation matrix represent the destinations of the two opposite corners, (1,0) and (0,1), of the initial, undeformed square, respectively.





Dilatation (equation 7) is given by

$$1 + \Delta = ad - bc$$

The rotational component of deformation is given by

$$\tan \omega = (b - c) / (a + d)$$

#### ***From a strain ellipse to the deformation matrix***

The long and short axes of the strain ellipse are defined by the principal stretches  $S_1$  and  $S_3$ , respectively (see equations 3 and 8). Define  $\theta$  the initial clockwise angle from the x-axis to  $S_1$  and  $\theta'$  this angle after deformation ( $\theta' = \theta + \omega$ ). Then:

$$a = S_3 \sin \theta' \sin \theta + S_1 \cos \theta' \cos \theta$$

$$b = S_3 \sin \theta' \cos \theta - S_1 \cos \theta' \sin \theta$$

$$c = S_3 \cos \theta' \sin \theta - S_1 \sin \theta' \cos \theta$$

$$d = S_3 \cos \theta' \cos \theta + S_1 \sin \theta' \sin \theta$$

#### ***From the deformation matrix to a strain ellipse***

The orientations of the strain axes are given by:

$$-\tan 2\theta' = 2(ac + bd) / (a^2 + b^2 - c^2 - d^2)$$

The principal strains are given by the two solutions to the following

$$\lambda = 0.5 \{ a^2 + b^2 + c^2 + d^2 \pm \sqrt{[(a^2 + b^2 + c^2 + d^2)^2 - 4(ad - bc)^2]} \}$$

#### ***Some special cases***

The matrix that does nothing

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A matrix that describes a simple dilation

$$\begin{bmatrix} 1 + \Delta & 0 \\ 0 & 1 + \Delta \end{bmatrix}$$

A rotation clockwise about the origin

$$\begin{bmatrix} \cos \omega & \sin \omega \\ -\sin \omega & \cos \omega \end{bmatrix}$$

A pure strain with strain axes parallel to x and y

$$\begin{bmatrix} 1 + \varepsilon_1 & 0 \\ 0 & -(1 + \varepsilon_1) \end{bmatrix}$$

A general pure strain

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

A simple shear parallel to  $x$

$$\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}$$

### *Displacement components*

A displacement vector  $\vec{d}$  links  $P_0$  to  $P_1$ . This vector is possibly not the exact path along which  $P_0$  has moved to  $P_1$ ; it only identifies the initial and final positions of the considered point, but may itself be the sum of several smaller vectors.

The displacement gradient tensor  $\mathbf{J}$  is very simply related to the deformation matrix:

$$\mathbf{J} = (\mathbf{D} - \delta) = \begin{bmatrix} a-1 & b \\ c & d-1 \end{bmatrix}$$

When a point  $x$  is multiplied by  $\mathbf{J}$ , the result is a vector describing the displacement of  $x$  (ie its change in location).

$$\mathbf{J}x_0 = x - x_0$$

The strain tensor representing the non-rotational part of the deformation is a variant of the displacement gradient tensor. Recall that a pure strain (no rotation of the strain axes) is represented by a symmetric matrix. We can design a symmetric matrix that comes close to representing the strain component of displacement, by averaging the two asymmetric terms  $b$  and  $c$ . This gives a matrix called the strain tensor  $\mathbf{E}$ :

$$\mathbf{E} = \begin{bmatrix} e_{xx} & e_{xy} \\ e_{yx} & e_{yy} \end{bmatrix} = \begin{bmatrix} a-1 & 1/2(b+c) \\ 1/2(b+c) & d-1 \end{bmatrix}$$

Every particle of a deformed body is associated with a displacement vector; collectively, all vectors define a **displacement field**.

General homogeneous deformation is a combination of three types of displacement: (1) Translation, (2) rotation and (3) stretch.

#### Translation

A field of parallel vectors with the same direction displaces all points of the body, which suffers no internal deformation, by the same distance:

$$x = x_0 + A$$

$$y = y_0 + B$$

$X_{ij} = a_{ij}x_j + u_i$  can be written in matrix form as  $\mathbf{X} = \mathbf{A}\mathbf{x} + \mathbf{U}$ . The constant values  $u_i$ , which are independent of position, represent the rigid body translation and are not considered further.

Since the displacement is uniform, the body suffers no internal deformation.

#### Rotation

All lines in a body uniformly change their orientation. Therefore, rotation does not affect the shape of the body. There are two cases:

Clockwise rotation:

$$x = x_0 \cos \omega + y_0 \sin \omega$$

$$y = -x_0 \sin \omega + y_0 \cos \omega$$

Anticlockwise rotation:

$$x = x_0 \cos \omega - y_0 \sin \omega$$

$$y = x_0 \sin \omega + y_0 \cos \omega$$

### *Vorticity*

Vorticity is a mathematical formulation related to the rate of rotation of fluid particles entrained in a swirl. The axis around which the particles rotate carries the vorticity vector whose magnitude equals the angular rate of rotation.

In pure shear, for every line rotating clockwise, there is a line rotating by precisely the same amount anticlockwise; the average rotation is zero. If deformation is non-coaxial, all material lines rotate around the intermediate strain axis. The average angular velocity of the rotating material lines is the **internal vorticity**, which is a vector parallel to the Y-axis. Internal vorticity is quantified with the **kinematic vorticity number**  $W$ . In non-compressible material, it is the ratio of angular rotation (in radians) to the extension along the X-axis. In two dimensions, the vorticity is also the cosine of the angle between the flow asymptotes. This measure of the amount of rotation compared to the amount of distortion yields a degree of non-coaxiality:

$W = 0$  for pure shear, which is coaxial deformation.

$W = 1$  for simple shear, which is rotational deformation.

Vorticity varies between 0 and 1 for general deformation.

#### Attention:

If the body is rotating during deformation, vorticity has two components: shear-induced vorticity and **spin**, which is the rotation of the strain axes.

### *Flow lines*

Describing flow lines is another useful method to display strain history.

In the coaxial case, flow along the strain axes is directly straight inward or outward. The strain axes act as flow asymptotes. They also coincide with the eigenvectors of the deformation matrix.

In non-coaxial flow, flow asymptotes do not usually coincide with the strain axes, even if they coincide with the eigenvectors of the deformation matrix. As deformation becomes more non-coaxial, the flow asymptotes get closer together until in the case of simple shear, there is only one asymptote. Typically, flow converges on one of the strain asymptotes. As strain becomes large (e.g. in a mylonite) the X-axis approaches this asymptote, and all the planar and linear fabric elements tend to converge on this line.

### *Displacement in three-dimensions*

The point  $P_0$  has initial, material coordinates  $(x_0, y_0, z_0)$ . After deformation,  $P_0$  has moved to  $P_1$  at spatial coordinates  $(x_1, y_1, z_1)$ .

Like for homogeneous deformation in two dimensions, the displacement  $P_0$  to  $P_1$  is mathematically a coordinate transformation:

$$x_1 = ax_0 + by_0 + cz_0$$

$$y_1 = dx_0 + ey_0 + fz_0$$

$$z_1 = gx_0 + hy_0 + iz_0$$

Which written in matrix form is:

$$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

where the deformation matrix  $\mathbf{D}$  for infinitesimal or homogeneous deformation is:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

This second-order tensor is symmetrical, which implies that there are three orientations, the three diagonal components, for which there is no shear strain. A vectorial quantity with the three Cartesian components  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  parallel to the  $x$ ,  $y$  and  $z$  coordinates, respectively, defines the displacement of the material point  $P$ . For infinitesimal strain problems, the displacement variation can be assumed to be linear and uninterrupted within the continuum body. Then, the vectorial components are differentiable with respect to the coordinates axes.

### Finite strain in 3D

We use and derive in three dimensions measures of strain analogous to those in two dimensions.

#### **Strain ellipsoid**

The size and orientation of the strain ellipsoid are given by the relative size and directions of the three mutually perpendicular principal strain axes  $X = (1 + \varepsilon_1) \geq Y = (1 + \varepsilon_2) \geq Z = (1 + \varepsilon_3)$ . These three axes are poles to the three principal planes of strain, which are the only three planes that suffer zero shear strain. The equation describing this ellipsoid is:

$$\frac{x^2}{(1 + \varepsilon_1)^2} + \frac{y^2}{(1 + \varepsilon_2)^2} + \frac{z^2}{(1 + \varepsilon_3)^2} = 1$$

Cross-sections of the strain ellipsoid are ellipses (but there can be circular cross-sections). In general, the way we determine the 3D strain ellipsoid is to saw up rocks to define surfaces on which we find 2D strain ellipses. Then recombine the ellipses into an ellipsoid.

#### **Strain tensor**

To specify the strain ellipsoid completely requires nine numbers organized in a 3 x 3 matrix.

The three values of  $X = (1 + \varepsilon_1) \geq Y = (1 + \varepsilon_2) \geq Z = (1 + \varepsilon_3)$

The plunge and trend of one strain axis after deformation

The rake of a second axis in the plane perpendicular to the first.

The plunge trend and rake for the three axes before deformation

The three-dimensional deformation gradient tensor is this 3 x 3 matrix.

For a non-rotational strain, the matrix is symmetrical and only 6 numbers are required because the orientations are the same before and after deformation.

#### **Rotation and strain in 3D**

Rotational strain in 3D is complex. The rotational component of deformation can be perpendicular to two of the strain axes and therefore parallel to the third. When this is the case, the strain is described as **monoclinic**.

However, the rotation axis can be different from any of the strain axes. In that case, there is no mirror plane and the strain is **triclinic**. Triclinic strain and especially progressive triclinic strain (where rotation and distortion occur concurrently but on different axes) is very tricky to work with and requires techniques beyond the scope of this course.

#### **Displacement vector field**

Since every point in a deforming body follows a displacement vector, there is a displacement vector field covering the entire body. We have complete knowledge of the deformation if we know this displacement vector field.

#### **Velocity field**

Describes the velocity and direction of motion of the particles for the increment considered

## *Flow lines*

### **Experimental rheology and strain**

Constitutive equations describing the ductile flow of rocks relate stress to the strain rate and not to the finite strain. This has important implications. Deformed rocks display a finite state of strain, which is the sum of all deformations these rocks have experienced. Since the strain rate essentially is the incremental strain per unit time and since finite strain ellipsoids do not provide any information about the deformation path, i.e. the incremental strain ellipsoids, ductile deformation structures cannot be directly interpreted in terms of stress orientation. In short, assuming that the maximum shortening direction of finite strain is parallel to the maximum compressive stress is a major blunder.

### **Conclusion**

The strain is the change in the shape of a body because of deformation due to applied stresses. A strain involves volume changes (dilation or compaction), length changes (extension or shortening) and changes of angle (shear strain). Three mutually perpendicular principal strain axes, typically  $X > Y > Z$  characterize strain states. The geologist analyzes the long-term cumulative effect of deformation, which is the finite strain. The strain analysis is concerned with the estimation, at various points in a rock body, of the form and orientation of the finite strain ellipsoid. Different deformation mechanisms can result in similar geometrical forms.

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